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A COMPARISON OF THE FORMULAS OF HELMHOLTZ AND NERNST FOR THE ELECTROMOTIVE FORCE OF CONCENTRATION CELLS.

By Henry S. Carhart.

Temperature Coefficients.

An instructive series of relations, expressing the several quantities as a function of the temperature, will serve as a fitting point of departure for this paper. These relations are the following:

$P = P_0(1 + at)$, for the pressure of a gas at constant volume.

$p = p_0(1 + at)$, for osmotic pressure.

$R = R_0(1 + at)$, for the resistance of pure metals.

$E = E_0(1 + at)$, for electrolytic thermo-electromotive force of infinitely dilute solutions.

The coefficient a has the same value in all four equations. It is $1/273$ or 0.00366 very approximately, at least for the pressure of an ideal gas, most nearly represented by hydrogen; for the resistance of pure platinum; and for the E. M. F. of any concentration cell with solutions indefinitely dilute. It is an interesting inquiry as to what is the common relationship indicated by the common constant a equal to $1/273$.

The differential coefficient of pressure with respect to temperature from the first equation is $dP/dT = P_0 a$; from which we have $a = 1/P_0 \cdot dP/dT$ as the most general expression for the temperature coefficient, with corresponding expressions for the other three relations.

If now P, for example, is zero at the absolute zero of tempera-

ture, then $0 = P_0(1 - 273a)$ and $a = 1/273 = 1/T_0$. Substitute this value of a in the equation above, and $dP/dT = P_0/T_0$. Whence $P_0 = T_0 \cdot dP/dT$. Similarly in general, $P = T \cdot dP/dT$; that is, *the pressure is proportional to the temperature on the absolute scale.* Corresponding expressions may be derived for the other quantities. If R is zero at the absolute zero of temperature, then $R = T \cdot dR/dT$ and $a = 1/T_0$. So also if E is zero at the zero of the absolute scale, then $E = T \cdot dE/dT$ and a for that relation is also $1/T_0$.

If in each of the four relations the quantity represented becomes zero at the zero of the absolute scale, on the assumption that the law holds down to that point, then the temperature coefficient a is $1/T_0$. Each equation may then serve to define and evaluate the absolute zero; and if this zero denotes anything real, such as the zero of heat energy, then the T_0's for the several equations may be expected to coincide. As a matter of fact, the coefficient a is practically the same in all four equations, and the corresponding absolute zeros are therefore approximately identical.

In the case of pure platinum Dewar and Fleming have extended experimentally the straight line graph connecting its resistance and temperature so far down the scale toward absolute zero that they are able to say that it runs nearly through $-273°$ C. The common constant a means that gas pressure (volume constant), osmotic pressure, electrical resistance, and electromotive force (under definite conditions) are all proportional to the absolute temperature.

If R has a positive value at the absolute zero, as it has in the case of alloys, then

$$R = R_{-273} + T\frac{dR}{dT}.$$

R_0 is then greater than $T_0 \cdot dR/dT$ and a is less than $1/T_0$. So also E at the absolute zero may have either a positive or a negative value, and then

$$E = E_{-273} + T\frac{dE}{dT}.$$

THE HELMHOLTZ EQUATION.

The famous Helmholtz equation for the electromotive force of a voltaic cell is

$$E = \frac{H}{nF} + T\frac{dE}{dT},$$

corresponding with the last equation above. In the Helmholtz equation H is the heat of reaction or of dilution, expressed in joules; n is the valence of the cation; and F is the number of coulombs transported by a gram-equivalent of the cation.

The term $T \cdot dE/dT$, proportional to the absolute temperature, is purely thermo-electric. It is now fully established and agreed that the coefficient dE/dT is the algebraic sum of the electromotive forces per degree arising at the several thermal junctions in the cell. If this coefficient is positive, then the cell converts some of the heat of the cell and its surroundings into the energy of an electric current; if it is negative, the electrical energy given out is less than the heat of reaction and dilution, and the cell heats. In the former case heat is absorbed and utilized; in the latter it is generated and wasted.

The second term of the Helmholtz equation denotes then thermo-electromotive force, and so does any other form of expression for electromotive force which is proportional to absolute temperature.

THE NERNST EQUATION FROM THERMO-ELECTROMOTIVE FORCE.

The curve in the accompanying figure exhibits the relation between the concentration of the electrolyte ($CdSO_4 + H_2O$) and the

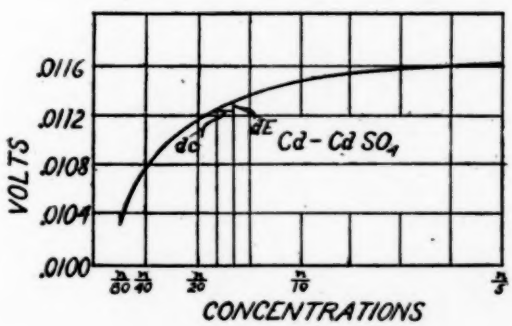

Fig. 1.

electromotive force per degree C. The electrodes were the same cadmium amalgam in every case, the only variation being the con-

centration of the electrolyte.[1] From this curve one may write the equation connecting electromotive force and concentration as follows:

$$dE = bT\frac{dc}{c}.$$

In this equation b is a proportionality factor and c is the concentration. Integrating,

$$E = bTlnc + \text{a constant}.$$

But when T is zero E is zero, and therefore the integration constant is zero. Therefore $E = bTlnc$ (ln is the natural logarithm). This expression is the potential difference arising at a single electrode. For a cell with different concentrations of the same electrolyte at the two electrodes, the E.M.F. is the difference between the two electrode potentials, or

$$E = bTlnc_2 - bTlnc_1 = bTln\frac{c_2}{c_1}.$$

It will be recognized that this equation agrees in form with that of Nernst for the E.M.F. of a concentration cell:

$$E = \frac{R}{nF}Tln\frac{c_2}{c_1}.$$

The Nernst equation, or its equivalent above, is only a variation of the second term of the Helmholtz equation; for the differential coefficient of E with respect to T from the first equation above is

$$\frac{dE}{dT} = bln\frac{c_2}{c_1}, \quad \text{and} \quad b = \frac{dE}{dT} \div ln\frac{c_2}{c_1}.$$

Substitute this value of b in the equation for E, and we have

$$E = T\frac{dE}{dT}.$$

The Nernst equation is therefore only another form of expression for that part of the E.M.F. of a concentration cell arising from electrolytic thermo-electromotive force; and it gives the correct value for the E.M.F. of such a cell only for infinite dilution, or when the

[1] Trans. Amer. Electrochemical Soc., Vol. II., p. 126; also PHYS. REV., Vol. XXV., p. 357.

heat of dilution is negligible. When there is no heat of dilution, the first term of the Helmholtz equation is zero and only the term expressive of thermo-electromotive force remains.

The Research of Richards and Forbes.

These conclusions find ample justification in two directions:

1. In the research of Richards and Forbes, published by the Carnegie Institution of Washington, and entitled "Energy Changes Involved in the Dilution of Zinc and Cadmium Amalgams."[1]

2. In the preliminary results of a research now in progress in my laboratory under the skilled hand of Mr. F. J. Mellencamp. In this research the observed and computed electromotive forces are compared for concentration cells with solutions of any degree of dilution.

The research of Professor T. W. Richards and Mr. Forbes is a thorough, careful and admirable piece of work. They measured the electromotive force of concentration cells in which the difference of concentration was in the amalgams composing the electrodes, an electrolyte of one concentration only being employed. The heat of dilution of the amalgams was measured directly by a special calorimeter. The dilution of a 0.9 per cent. zinc amalgam with an equal weight of mercury *absorbed* heat at the rate of 52 joules per gram-molecule of zinc; that is, the dilution of zinc amalgam has a cooling effect. The heat of dilution of a 3 per cent. cadmium amalgam was so small as to defy direct measurement, but it is probably slightly positive; that is, the dilution of cadmium amalgam generates heat.

The authors say on page 60: "One of the striking facts in relation to cadmium amalgam is the fact that its heat of dilution is so small as to be negligible. Therefore, the equation of Helmholtz reduces practically to the form

$$E = T\frac{dE}{dT},$$

making the thermodynamics of the problem as simple as possible."[2]

Now under these conditions the temperature coefficient a reduces to $1/T_0$. For since

[1] Publication No. 56, 1906.

[2] I have changed the nomenclature to agree with that commonly employed by physicists.

$$E_0 = T_0 \frac{dE}{dT},$$

the coefficient

$$\frac{1}{E_0}\frac{dE}{dT} = \frac{dE}{dT} \div T_0 \frac{dE}{dT} = \frac{1}{T_0}.$$

The agreement of this coefficient with that of gas pressure signifies merely that the electromotive force is very approximately proportional to T, the absolute temperature, for then a always equals 1/273 or 0.00366.

The accuracy of the work of Richards and Forbes is attested by the fact that the measured temperature coefficient for the cadmium amalgams is very close to 0.00366. On this point I quote the following:

"It becomes now a matter of great interest to compare this change of potential with the requirements of the gas law, by comparing the temperature coefficient with the temperature-pressure coefficient of a perfect gas over the same range of temperature.

"The following table gives the temperature coefficients referred to the observed potentials at zero:

2–4, from 30° to 15.2°.

$$\frac{1}{E_0}\frac{\Delta E}{\Delta T} = \frac{0.001659}{14.8 \times 0.030826} = 0.00364$$

2–4, from 15.2° to 0.0°.

$$\frac{1}{E_0}\frac{\Delta E}{\Delta T} = \frac{0.001719}{15.2 \times 0.030826} = 0.00367$$

1–4, from 30° to 15.2°.

$$\frac{1}{E_0}\frac{\Delta E}{\Delta T} = \frac{0.002578 \times 288.2}{14.8 \times 0.050332 \times 273} = 0.003665$$

1–3, from 30° to 15.2°.

$$\frac{1}{E_0}\frac{\Delta E}{\Delta T} = \frac{0.00177 \times 288.2}{14.8 \times 0.03413 \times 273} = 0.00366$$

Average 0.003656

"The agreement is surprisingly good; within the limit of accuracy of the measurement the increase of the potential with increase

of temperature is identical with the increase of pressure of a perfect gas."

These remarks and others of similar import lead one to the conclusion that Professor Richards regards this agreement of the electromotive force coefficient with the pressure coefficient, both with respect to temperature, as a proof that the gas law, $pv = RT$, applies to a concentration cell. On the contrary it appears from this discussion that this agreement of coefficients means only that the electromotive force of a concentration cell, with negligible heat of dilution, is proportional to the absolute temperature.

It is particularly unfortunate that these investigators did not determine the temperature coefficient for pairs of zinc amalgams, for it is certain that they would have found a value greater than 0.00366 except in the case of very dilute amalgams. They did *calculate* one temperature coefficient for zinc amalgams and found it to be 0.00379.

The identity of the temperature coefficient with $1/273$ is not a satisfactory proof even that the heat of dilution is negligible, for the electromotive force due to heat of dilution affects only the value of E_0 in the expression for the temperature coefficient, $1/E_0 \cdot dE/dT$, and a moderately small change in the denominator of a small fraction produces a nearly negligible change in the value of the fraction itself.

A better criterion to apply as a test of the presence or absence of heat of dilution is to compare the measured values of electromotive force with those derived from the expression $T \cdot dE/dT$, or its equivalent in the form of the Nernst equation. This method is beautifully illustrated in Fig. 2, copied from the paper of Richards and Forbes. The logarithms of the concentration ratios are plotted as abscissas and the deviations of the observed electromotive forces from the theoretical values given by the Nernst equation as ordinates. It will be observed that the electromotive forces for the zinc amalgams are smaller than the values derived from the Nernst formula, while those of the cadmium amalgams are larger. Moreover, for very dilute amalgams the curves both approach indefinitely near the theoretical value of the Nernst formula. The zinc amalgams show much larger deviations from the computed values than do the cadmium amalgams.

Both of these deviations are completely and satisfactorily explained by the appreciable negative heat of dilution of zinc amalgams and the very small positive heat of dilution of the amalgams of cadmium. When the heat of dilution is negative, the first term of the Helmholtz equation is also negative, and the E.M.F. of the

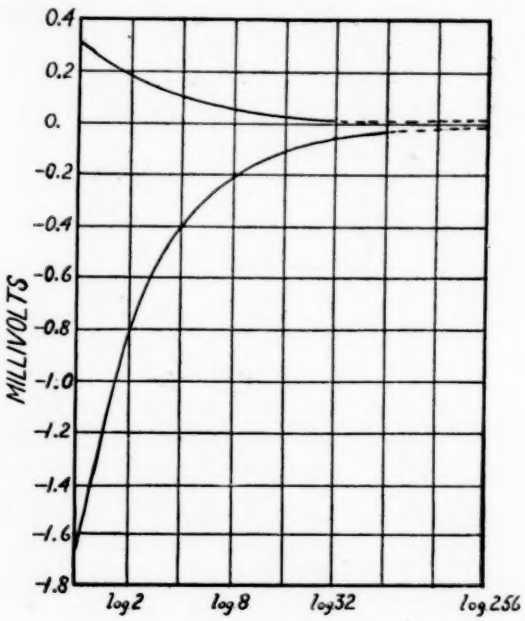

Fig. 2.

cell is less than $T \cdot dE/dT$. When the heat of dilution is positive, the first term of the Helmholtz equation is positive, and the E.M.F. of the cell is greater than $T \cdot dE/dT$. Since the Nernst equation is only the equivalent of the second term of the Helmholtz equation, the deviations of the curves in Fig. 2 from the straight line, which they both approach asymptotically, are completely explained by the first term of the Helmholtz equation, derived from the heat of dilution.

It is therefore useless to attempt to correct the Nernst formula so as to make it cover the case of concentration cells in which there is heat of dilution to affect the value of the E.M.F. It applies only to cells with infinite dilutions. The second term only of the Helmholtz equation is proportional to the absolute temperature. The first term,

while not necessarily independent of temperature, is not proportional to it and cannot therefore be included in the Nernst equation or in any modification of it, unless another term is added. Richards and Forbes appear to miss entirely the cause of the deviations of the observed electromotive forces from the values computed by the Nernst equation, and their speculations relative to the cause of these deviations are thus of little practical value. The reason for them lies directly on the surface, and it has probably been missed because of the point of view from which the research was carried out.

The Research of Carhart and Mellencamp.

The preceding conclusions, involving a comparison of the Helmholtz and Nernst equations, are completely justified by the measurements on a series of concentration cells made with solutions of electrolytes whose heats of dilution are known. In most of these cases the concentrations are such that the heats of dilution are relatively large, and the first term of the Helmholtz equation can not be neglected. No better justification of the Helmholtz equation than these results has ever been presented.

Whenever the heats of dilution of the solutions in the two legs of a concentration cell differ, the difference is available as energy to produce electromotive force, represented by the first term of the Helmholtz equation. The electrode in the more dilute solution is the anode. Let us suppose the quantities of the two solutions so large that the entrance of a gram-equivalent of the metal at the anode and the removal of an equal quantity from the solution about the cathode do not change the concentrations of the solutions. Then when a gram-equivalent of the metal at the anode becomes a salt, it is diluted down to the same concentration as that of the solution surrounding the anode; at the same time a gram-equivalent of the same metal is removed from the more concentrated solution surrounding the cathode. This latter operation absorbs less energy than is derived from the former, if the heat of dilution is positive; and this surplus energy is available to produce electromotive force.

To illustrate: One gram-molecule of $ZnSO_4.20H_2O$ diluted to $ZnSO_4.50H_2O$ generates 318 calories of heat (Thomsen). Also, $ZnSO_4.20H_2O$ diluted to $ZnSO_4.400H_2O$ generates 400 calories.

Hence, $ZnSO_4.50H_2O - ZnSO_4.400H_2O$ is equivalent to the difference between 400 and 318, or 82 calories. If a cell is set up with these two solutions, the E.M.F. corresponding to this difference in the heats of dilution is

$$\frac{82 \times 4.186}{2 \times 96{,}550} = 0.00178 \text{ volt.}$$

The coefficient dE/dT of a cell set up with electrodes of liquid zinc amalgam in the above solutions is positive and equal to 0.0000528, as a mean of several determinations. Hence, for 25° C. the term $T \cdot dE/dT$ is $298 \times 0.0000528 = 0.01573$. Then

$$E = 0.00178 + 0.01573 = 0.0175 \text{ volt.}$$

The observed E.M.F. at 25° was 0.0171.

In the following table E_h is the E.M.F. due to the heat of dilution, and $T \cdot dE/dT$ is the E.M.F. derived from the heat of the surroundings, except in the last example where it is negative.

Solutions.	$\frac{dE}{dT}$	$T\frac{dE}{dT}$	E_h	Comp. E.M.F.	Obs. E.M.F.
$ZnSO_4$. 50 H_2O $ZnSO_4$. 400 H_2O	0.0000528	0.01573-25°	0.00178	0.0175	0.0171
$ZnSO_4$. 100 H_2O $ZnSO_4$. 400 H_2O	0.000032	0.00954-25°	0.00072	0.0103	0.0101
$ZnSO_4$. 50 H_2O $ZnSO_4$. 100 H_2O	0.0000174	0.00518-25°	0.00106	0.0062	0.0062
$ZnSO_4$. 50 H_2O $ZnSO_4$. 200 H_2O	0.0000344	0.01025-25°	0.00145	0.0117	0.0116
$CdSO_4$. 50 H_2O $CdSO_4$. 400 H_2O	0.0000188	0.00557-23°	0.01080	0.0164	0.0164
$CdSO_4$. 50 H_2O $CdSO_4$. 200 H_2O	0.000009	0.00246-0°	0.00847	0.0109	0.0108
$CdSO_4$. 100 H_2O $CdSO_4$. 400 H_2O	0.0000155	0.00423-0°	0.00604	0.0103	0.0099
$CuSO_4$. 60 H_2O $CuSO_4$. 200 H_2O	0.0000248	0.00739-25°	0.00251	0.0099	0.0098
$ZnCl_2$. 21.9 H_2O $ZnCl_2$. 386 H_2O	−0.000104	−0.02839-0°	0.10100	0.0726	0.0751

The last two columns show extremely good agreement between the computed and observed values of the electromotive forces.

The last case furnishes a specially instructive illustration of the principles involved, because the temperature coefficient is negative, as in most voltaic cells. The E.M.F. of this cell is accordingly less than the value derived from the heat of dilution. The concentrations of the solutions of zinc chloride were determined by titration, and the corresponding heats of dilution by interpolation by means of a smooth curve plotted from recorded measurements.

In the case of the sulphates, the E.M.F. due to heat of dilution runs from 7 to 78 per cent. of the whole E.M.F. of the cell.

The temperature coefficients of the above concentration cells, referred to the observed electromotive force at zero, are the following in the order of the table: 0.00335, 0.00344, 0.00301, 0.00320, 0.00118, 0.00083, 0.00156, 0.00270, 0.00138. These coefficients do not look very much like a confirmation of the requirements of the gas law. The temperature coefficient is not 0.00366 simply because the E.M.F. of these cells is not in any case proportional to the absolute temperature.

UNIVERSITY OF MICHIGAN,
December, 1907.

A VARIATION OF THE BJERKNES ANALOGY BETWEEN HYDRODYNAMIC AND ELECTRIC OR MAGNETIC FIELDS.

BY A. P. WILLS.

I. INTRODUCTORY.

1. *The Bjerknes Analogy.*

IN the hydrodynamic analogy to electric or magnetic fields developed by Professor V. Bjerknes,[1] following out and extending the ideas of his late father, Professor C. A. Bjerknes, he makes the velocity of the fluid correspond to the electric or magnetic flux, the specific momentum (momentum per unit volume) to the electric or magnetic field intensity, and the specific volume to the electric or magnetic inductivity.

Employing this correspondence, he shows that there is a complete geometric analogy between the vectors representing the velocity and specific momentum in the hydrodynamic field and the vectors representing the flux and the field intensity in the electric or magnetic field, the analogy extending to stationary phenomena. Expressed in other words, he finds for stationary (including static) phenomena that the relationship of the vectors representing the velocity and specific momentum is expressed by a system of equations formally identical with that representing the relationship of the vectors corresponding to the flux and field intensity in the electric or magnetic field. The dynamic analogy found is not quite so complete; for although the expression for the resultant mechanical force acting upon a body in the hydrodynamic field can be given a form in which the individual terms are of the same type as the corresponding expression for the mechanical force acting upon a material body in an electric or magnetic field, the signs of these

[1] Reference may be made in this connection to the following sources: V. Bjerknes, Vorlesungen uber hydrodynamische Fernkräfte, nach C. A. Bjerknes' Theorie, Leipzig, 1900–1902. V. Bjerknes, Fields of Force — a course of lectures delivered at Columbia University in December, 1905.

terms are opposite. Various forms of expression are found for the elementary mechanical force acting per unit volume at any point of the fluid, and among these there is one which has, except for the reversed sign, precisely the analogous form to that given by Heaviside for the corresponding case in the electric or magnetic field.

The analogy which will be developed here, and which, for reasons which become obvious later, will be termed the change-of-state analogy, has many properties in common with the Bjerknes analogy, and on this account a brief summary of some of the cardinal features of the latter will now be given.

Corresponding to the ether, there is in the Bjerknes analogy the *fundamental fluid*, and corresponding to the material bodies of different inductivities there are the *fluid bodies*. The *fluid bodies* are in general of different density than the *fundamental fluid*, in which they are supposed to be immersed, and in general the properties of motion of the fluid constituting a *fluid body* are quite different from those associated with the *fundamental fluid*. Difference in density of course implies difference in specific volume, and this latter quantity corresponds to electric or magnetic inductivity. In accordance with this then, a *fluid body* of greater density than the *fundamental fluid* would in the magnetic case correspond to a diamagnetic body, while a *fluid body* of less density than the *fundamental fluid* would correspond to a paramagnetic body. (Just the reverse is the case in the change-of-state analogy.) An important difference between the Bjerknes analogy and the change-of-state analogy lies in the fact that in the Bjerknes analogy a *fluid body* is always constituted of the same particles of fluid, and this is also true for the *fundamental fluid*; but in the change-of-state analogy a particle of fluid may at one instant be a constituent of the *fundamental fluid* and later a constituent of some *fluid body*.

Referring still to the Bjerknes analogy, we have next to notice the conditions which have to be fulfilled at a surface of separation. If the analogy in this respect is to hold, the normal components of the velocity and the tangential components of the specific momentum must pass through the surface continuously (it being supposed that in the electric or magnetic case there is no true surface charge). The former of these conditions requires in general that the surface

of a fluid body shall be constantly changing its configuration and position in space, so that a *fluid body* would not strictly correspond to a material body in the electric or magnetic field, since the latter remains at rest. This difficulty is surmounted by supposing the fluid motion to be of a vibratory nature, consisting of small vibrations about a mean configuration, so that the fluid bodies may appear to be at rest. The hydrodynamic vectors which correspond to the electric and magnetic vectors, and which describe the vibratory motion, are mean values of the actual periodically varying vectors. These mean vectors satisfy the boundary conditions; but owing to a peculiar property of the fluid motion, the analogy only extends to static electric or magnetic fields. In the change-of-state analogy the boundary conditions do not introduce the difficulty just considered.

We have noticed here only some of the chief features of the Bjerknes analogy, but in developing the change-of-state analogy we shall frequently recur to it.

2. *The Change-of-State Analogy.*

As mentioned above, the change-of-state analogy has much in common with the Bjerknes analogy. The mathematical developments pertaining to each are very similar. In thinking and working on the new form of analogy I have been much benefited by many conversations with Professor Bjerknes, also by his letters and writings, and I have pleasure in acknowledging my indebtedness to him in this connection. I have made free use of the methods of Professor Bjerknes, and I have also employed, in large measure, his notation, which is most admirably adapted to the purposes in view.

In the change-of-state analogy the scheme of correspondence between the hydrodynamic quantities, on the one hand, and the electric or magnetic quantities, on the other, is the following:

HYDRODYNAMIC.	ELECTRIC OR MAGNETIC.
Velocity	Field Intensity
Specific momentum	Flux
Density	Inductivity.

It will be remembered that in the Bjerknes analogy velocity corresponds to flux, specific momentum to field intensity, and the reciprocal of density (specific volume) to inductivity. By the term specific momentum is to be understood the momentum per unit volume.

We have now to imagine the fluid system which is to correspond in the change-of-state analogy to a stationary electric or magnetic system. The fluid system which we consider consists of a *fundamental fluid* corresponding to the ether, and of *fluid bodies* corresponding to material bodies in the electric or magnetic field. In cases where it is not necessary to discriminate between the *fundamental fluid* and the fluid of a *fluid body* we shall simply use the term — the fluid. The *fundamental fluid* is supposed to be of infinite extent, being bounded by the stationary closed surfaces of the regions containing the *fluid bodies* and the surface of the infinite sphere. A *fluid body* consists of fluid within one of these closed surfaces, and differs from the *fundamental fluid* in that it is compressible, and that the characteristics of motion of the fluid in it are quite different and not so simple as in the *fundamental fluid;* furthermore the density of the fluid in a *fluid body* may be variable from point to point and in general different from that of the *fundamental fluid.* Fluid may cross the surface of the region bounding a *fluid body* (the surface of the *fluid body*). Upon doing so it is supposed to suddenly change its state. Thus if it is fluid which enters the region of a *fluid body* it suddenly loses the state of motion characteristic of the *fundamental fluid,* and acquires that which is characteristic of the *fluid body;* and associated with this change of state there is in general a change of density. In a similar way, fluid which escapes from the region of a *fluid body* loses the properties of the *fluid body,* and acquires those of the *fundamental fluid.* (It is not necessary, for the purposes in view, to specify the nature of the mechanism which would produce the changes at the surface of the *fluid body* which are supposed to occur.) From the preceding remarks it appears that a given particle of fluid may at one time be a constituent part of the *fundamental fluid,* and at another time a part of a *fluid body.* If in its wanderings it chances to cross the surface of a fluid body, it will suddenly change its state — lose certain properties and

acquire new ones. When a portion of fluid constitutes a part of the *fundamental fluid*, it is said to be in the *fundamental state*, and when it constitutes part of a fluid body it is said to be in the *body state*.

Boundary Conditions. — The two boundary conditions which have to be fulfilled in the change-of-state analogy are:

1. At any surface of separation the normal components of the specific momentum must be continuous, if it be supposed that no accumulation of mass occurs at the surface.

2. At any surface of separation the tangential components of the velocity must be continuous.

The principle of the conservation of mass requires that the first of these conditions be fulfilled. For, if there is to be no accumulation of mass at the surface of separation, the mass of fluid which arrives per unit area per unit time at one side of the surface must be equal to the corresponding mass which leaves the surface on the other side; and the expression of this fact requires the continuity of the normal components of the specific momentum.

With regard to the second boundary condition, it has simply to be remarked that the sudden change in density which is supposed to occur at the surface of separation does not implicitly require the assumption of an infinite tangential force, and hence we suppose none such to be present there; it follows then that there can be no finite change in the velocity tangential to the surface of separation as the fluid crosses the surface, or, in other words, that the tangential components of the velocity are the same on both sides of the surface.

A concrete illustration of the sort of change which is supposed to occur at the surface of a fluid body is furnished by the familiar phenomenon of condensation, or of evaporation.

Induced and Energetic Vectors. — In the theory of electricity and magnetism it is customary to consider the actual field intensity and the actual flux at any point as made up of two parts: (1) A part which is due to the electric or magnetic induced field; (2) a part which is due to the existence at the point in question of an intrinsic electric or magnetic source of some sort. Vectors corresponding to the first part are called induced vectors; and vectors correspond-

ing to the second part are called energetic vectors. Intrinsic polarization of a permanent magnet, sources of thermoelectric force, or of contact electric force, are examples of energetic sources. The energy of an electric or magnetic field is established primarily through energetic sources; and hence Professor Bjerknes has adopted the term — energetic vectors — for vectors relating to intrinsic sources. In a similar way, the actual hydrodynamic vectors may be considered as made up of induced and energetic parts.

Hydrodynamic Notation. — The notation employed in the change-of-state analogy to designate the actual, induced, and energetic vectors, and the density of the fluid is given in Table I.

TABLE I.

Quantity.	Actual.	Induced.	Energetic.
Sp. momentum	$\mathbf{C}$	$\mathbf{C}_i$	$\mathbf{C}_e$
Velocity	$\mathbf{c}_a$	$\mathbf{c}$	$\mathbf{c}_e$

Density of fluid $= \gamma$.

Among the six vectors given in Table I. there are four independent relations which constitute what are called the equations of connection and which will be given presently.

Elementary vector methods are employed in the mathematical work which follows, on account of the great saving of space which is thereby secured. The notation is essentially that used in Gibb's Vector Analysis.

II. DEVELOPMENT OF THE EQUATIONS FOR THE HYDRODYNAMIC SYSTEM IN THE CHANGE-OF-STATE ANALOGY.

In order to obtain a comprehensive idea of the nature and properties of the hydrodynamic system which is employed in the change-of-state analogy, we shall first investigate the properties of the hydrodynamic field. Afterwards the properties of this hydrodynamic field will be compared with those of the stationary electric or magnetic field.

Equations of Connection. — Among the vector quantities found in Table I. there is the following set of equations of connection:

$$(C_1) \qquad \begin{aligned} \mathbf{C} &= \mathbf{C}_i + \mathbf{C}_e, \\ \mathbf{c}_a &= \mathbf{c} + \mathbf{c}_e, \\ \mathbf{C}_i &= \gamma \mathbf{c}, \\ \mathbf{C}_e &= \gamma \mathbf{c}_e. \end{aligned}$$

Remembering that the actual specific momentum and the actual velocity are each made up of corresponding induced and energetic parts; and that the product of density into velocity gives the corresponding specific momentum, these equations are obviously true.

The Equation of Motion and the Equation of Continuity. — In the development of the properties of the hydrodynamic system which is employed in the change-of-state analogy, the equation for a perfect fluid and the equation of continuity furnish the starting point.

The equation of motion is

$$(1) \qquad \gamma \frac{d\mathbf{c}_a}{dt} = \mathbf{f} - \nabla p,$$

wherein $\mathbf{f}$ denotes the external force acting per unit volume of the fluid, p the pressure, and $-\nabla p$ the pressure gradient, γ the density, and $d\mathbf{c}_a/dt$ the actual acceleration of a particle of the fluid. The equation states that the kinetic reaction per unit volume of the fluid is equal to the sum of the forces per unit volume due to external agencies and to fluid pressure.

The equation (1) is not at all suggestive of the equations of the electric or magnetic field. But the equation is capable of development into a form which is appropriate to the purposes in view. This development will be given presently.

In writing the equation of motion in the form given, it is tacitly assumed, of course, that the fluid constituting the *fundamental fluid* or any *fluid body* has the properties of a perfect fluid.

To express the condition of conservation of mass we have the equation of continuity,

$$(2) \qquad -\frac{\partial \gamma}{\partial t} = \operatorname{div} \mathbf{C},$$

which states that the rate of decrease of the mass contained in a fixed element of volume is equal to the rate at which fluid leaves the element of volume.

In connection with equations (1) and (2), we have occasionally to make use of the Eulerian Operator, which relates the *individual* rate of change of a quantity with the *local* rate of change of the same quantity. This operator is expressed as follows:

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \mathbf{c}_a \nabla(\ ). \tag{3}$$

Proceeding now to the development of equation (1), we express, with the aid of equation (3), the left hand member as follows:

$$\gamma \frac{d\mathbf{c}_a}{dt} = \gamma \frac{\partial \mathbf{c}_a}{\partial t} + \gamma \mathbf{c}_a \nabla \mathbf{c}_a,$$

and referring to the first term on the right of this last expression, we may write

$$\gamma \frac{\partial \mathbf{c}_a}{\partial t} = \frac{\partial(\gamma \mathbf{c}_a)}{\partial t} - \mathbf{c}_a \frac{\partial \gamma}{\partial t},$$

or

$$\gamma \frac{\partial \mathbf{c}_a}{\partial t} = \frac{\partial \mathbf{C}_i}{\partial t} + \frac{\partial \mathbf{C}_e}{\partial t} + \mathbf{c}_a \operatorname{div} \mathbf{C},$$

with the aid of the equations of connection (C_1) and the equation of continuity (2); hence,

$$\gamma \frac{d\mathbf{c}_a}{dt} = \frac{\partial \mathbf{C}_i}{\partial t} + \frac{\partial \mathbf{C}_e}{\partial t} + \mathbf{c}_a \operatorname{div} \mathbf{C} + \gamma \mathbf{c}_a \nabla \mathbf{c}_a.$$

The last term on the right of this expression may be transformed as follows:

$$\begin{aligned} \gamma \mathbf{c}_a \nabla \mathbf{c}_a &= \gamma \mathbf{c}_a \mathbf{C}_a \nabla + (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C} \\ &= \tfrac{1}{2} \gamma \nabla \mathbf{c}_a^{\,2} + (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C} \\ &= \nabla \tfrac{1}{2} \gamma \mathbf{c}_a^{\,2} - \tfrac{1}{2} \mathbf{c}_a^{\,2} \nabla \gamma + (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C}; \end{aligned}$$

and introducing this value for $\gamma \mathbf{c}_a \nabla \mathbf{c}_a$ in the last expression for $\gamma d\mathbf{c}_a/dt$, we get

$$\gamma \frac{d\mathbf{c}_a}{dt} = \frac{\partial \mathbf{C}_i}{\partial t} + \frac{\partial \mathbf{C}_e}{\partial t} + \mathbf{c}_a \operatorname{div} \mathbf{C} + \nabla \tfrac{1}{2} \gamma \mathbf{c}_a^{\,2} - \tfrac{1}{2} \mathbf{c}_a^{\,2} \nabla \gamma + (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C}.$$

Upon inserting this value for $\gamma d\mathbf{c}_a/dt$ in equation (1), there results

$$\frac{\partial \mathbf{C}_i}{\partial t} + \mathbf{c}_a \operatorname{div} \mathbf{C} + \nabla \tfrac{1}{2} \gamma \mathbf{c}_a^{\,2} - \tfrac{1}{2} \mathbf{c}_a^{\,2} \nabla \gamma + (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C} + \frac{\partial \mathbf{C}_e}{\partial t} = \mathbf{f} - \nabla p. \tag{4}$$

In the preceding discussion we have not specified the induced and energetic vectors beyond the requirements of equation (C_1). In the following paragraph further specification of these quantities will be given in such a way that the geometric and dynamic properties of the fluid motion may be shown in a form which will prove suitable for the analogy with the electric or magnetic field.

Further Specification of the Induced and Energetic Vectors. — It is quite allowable to impose any conditions we like upon the induced and energetic vectors which are not in conflict with equations (C_1). Let, then, the induced specific momentum satisfy the equation

$$\frac{\partial \mathbf{C}_i}{\partial t} = -\nabla(p + \tfrac{1}{2}\gamma \mathbf{c}_a^2). \tag{5}$$

From equations (4) and (5) it is obvious that the energetic specific momentum must satisfy the equation

$$\frac{\partial \mathbf{C}_e}{\partial t} = \mathbf{f} - \mathbf{c}_a \operatorname{div} \mathbf{C} + \tfrac{1}{2}\mathbf{c}_a^2 \nabla\gamma - (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C}. \tag{6}$$

3. *Geometric Properties of the Hydrodynamic Field.*

By taking the curl of both members of equation (5) and then reversing the order of the operations of differentiation and curl on the left, it is seen that

$$\frac{\partial}{\partial t}(\operatorname{curl} \mathbf{C}_i) = 0. \tag{7}$$

Integration of equation (7) gives

$$\operatorname{curl} \mathbf{C}_i = -\mathbf{K}, \tag{8}$$

where $\mathbf{K}$ is a quantity which is independent of the time, and called by Professor V. Bjerknes the *density of the dynamic vortex*. The condition imposed upon the induced motion through equation (5) is such as to require the dynamic vortex density at any point to be invariable; and hence, if the dynamic vortex density is initially zero at all points of a given region, no dynamic vortex motion will ever exist in this region.

It is an interesting fact that the property of conservation of the dynamic vortex motion is likewise present in the hydrodynamic system employed in the Bjerknes form of the analogy.

For purposes of the analogy which we have in view it is important to introduce a quantity called by Professor Bjerknes the *kinematic vortex density*. This is simply the negative of the curl of the induced velocity and is designated by $\mathbf{k}$; thus

$$(C_2) \qquad \text{curl } \mathbf{c} = -\mathbf{k}.$$

The kinematic vortex, unlike the dynamic, is not subject to conservation in general; but if the motion throughout the system is steady, then the kinematic vortex motion will be conserved. It is an important quantity in that it corresponds to current in the electric or magnetic field.

Another important quantity with which we shall have to deal is the divergence of the specific momentum, designated by H; thus

$$(C_3) \qquad \text{div } \mathbf{C} = H.$$

It is important in that it is the quantity which corresponds to the density of electricity or of magnetism in the electric or magnetic field. From the equation of continuity (2) it is evident that it represents the local time rate of decrease of the density of the fluid at any point.

An important property of the energetic motion is easily established for a region throughout which certain conditions are fulfilled. These conditions are the following: (1) The external forces shall vanish, (2) the fluid shall be homogeneous, (3) the fluid shall be incompressible, (4) the dynamic vortex density shall vanish, (5) the energetic velocity shall be initially zero.

Condition (4) is equivalent to requiring that initially no dynamic vortex motion be present in the region, on account of the property of conservation of the dynamic vortex density which we have noted above. Furthermore, since the density of the fluid is constant, according to conditions (2) and (3), and since the dynamic vortex density in this case will be proportional to the kinematic, the density of the fluid being the factor of proportionality, it follows that condition (4) is equivalent to requiring the kinematic vortex density to vanish throughout the region. On account of the first four conditions, it is easily seen that equation (6) for the region in question reduces to

$$\frac{\partial \mathbf{C}_e}{\partial t} = -(\text{curl } \mathbf{c}_e) \times \mathbf{C};$$

from which it appears that if the energetic velocity be initially equal to zero throughout the region, then the energetic specific momentum must be always equal to zero throughout the region. In making this deduction it has to be observed that the energetic specific momentum is simply proportional to the energetic velocity, the density of the fluid being a constant factor of proportionality.

As a result of the investigation with which we have just been occupied, it is evident that no inconsistency will be involved in requiring the fluid in a given region to satisfy the following conditions: (1) The fluid shall be homogeneous, (2) the fluid shall be incompressible, (3) the kinematic vortex motion shall vanish, (4) the energetic specific momentum shall vanish.

The properties of the induced and the energetic motions which we have touched upon here will be of service in defining the properties of the *fundamental fluid* and the *fluid bodies*.

4. *The Fundamental Fluid and the Fluid Bodies.*

In accordance with the introductory remarks at the beginning of this paper, it will be remembered that the fluid system which is considered is constituted as follows: There are supposed to exist certain fixed regions bounded by closed surfaces, within which the fluid motion is in general different from that in the fluid which surrounds these regions and which is supposed to have the same properties throughout and to extend to infinity. These regions are referred to as the *fluid bodies*, and the surrounding fluid is called the *fundamental fluid*. It should be recalled that the *fundamental fluid* in crossing the surface of a body inwards experiences in general a sudden change in density, in fact loses the characteristic properties of the *fundamental fluid*; and in a similar way fluid which escapes from a region defining a *fluid body* suddenly acquires the properties of the *fundamental fluid*. According to this conception, the region containing the *fundamental fluid* and the regions defining the *fluid bodies* are filled with the same fluid, the difference in the regions arising from the differences in the character of the motion and the physical state of the fluid when it is in what we may call the fundamental region and when it is within a region defining a body. A *fluid body*, then, is a region in which the density of the fluid is in

general different from that of the fluid in the fundamental region, and whose state of motion is also different. Whether a given part of the fluid constitutes a part of the *fundamental fluid* or of a *body*, depends on its *state* at the time considered.

The conditions characterizing the state of the *fundamental fluid* may be expressed by the following group of equations, in accordance with the results of the preceding section:

$$(C_4) \qquad \begin{array}{ll} (1)\ \gamma = \gamma_0 = \text{const.}, & (2)\ H = 0, \\ (3)\ \mathbf{k} = 0, & (4)\ \mathbf{C}_e = 0. \end{array}$$

These equations respectively require that the *fundamental fluid* be homogeneous; that it be incompressible; that it support no kinematic vortex motion; and that throughout it the energetic specific momentum shall always vanish. These conditions being fulfilled for the *fundamental fluid*, it will never experience any external force (see eq. (6)).

It will presently appear that a set of equations exactly corresponding to equations (C_4) is used in characterizing the ether for the case of the stationary electric or magnetic field.

A *fluid body* differs from the *fundamental fluid* in that none of the restrictive conditions expressed by equations (C_4) is impressed upon it.

5. *The Dynamical Properties of the Fluid Motion.*

In considering the dynamical properties of the fluid system employed in the change-of-state analogy it turns out that the character of the fluid motion is more general than is required. The condition is therefore introduced that the energetic specific momentum shall at all points of the system be locally conserved. This requirement is expressed by writing

$$\frac{\partial \mathbf{C}_a}{\partial t} = 0. \tag{9}$$

In order that this condition may be fulfilled, it is necessary, on account of equation (6), that

$$\mathbf{f} = \mathbf{c}_a \operatorname{div} \mathbf{C} - \tfrac{1}{2}\mathbf{c}_a^2 \nabla\gamma + (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C}.$$

This equation gives the force per unit volume exerted by agencies exterior to the fluid system upon the fluid system; and $-\mathbf{f}$ $(= \mathbf{f}_h'$,

say) will be the corresponding force exerted by the fluid against the exterior agencies. We thus have

$$(C_5) \qquad \mathbf{f}_h' = -\mathbf{c}_a \operatorname{div} \mathbf{C} + \tfrac{1}{2}\mathbf{c}_a^2 \nabla\gamma - (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C},$$

expressing the force per unit volume of fluid against exterior agencies.

An expression for the *resultant force* acting upon any *fluid body* is obtained through volume integration of equation (C_5). If $d\tau$ denote an element of volume and $\mathbf{F}_h'$ the resultant force, then

$$(C_6) \qquad \mathbf{F}_h' = -\int \mathbf{c}_a \operatorname{div} \mathbf{C} d\tau + \int \tfrac{1}{2}\mathbf{c}_a^2 \nabla\gamma d\tau - \int (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C} d\tau,$$

the integrations being extended throughout the whole body.

6. *Specifications which Uniquely Determine the Hydrodynamic Field.*

In order to preclude the possibility of ambiguity in the comparison of the hydrodynamic field with the stationary electric or the stationary magnetic field, it seems advisable at this point to establish the following theorem:

The hydrodynamic field is uniquely determined when the distributions of the divergence of the actual specific momentum, of the curl of the induced velocity, and of the energetic specific momentum are given.

To establish this theorem we consider first the kinetic energy of the fluid. The kinetic energy of the whole fluid system is represented by either of the equivalent equations

$$(10) \qquad \Phi = \int \tfrac{1}{2}\gamma \mathbf{c}_a^2 d\tau,$$

$$(11) \qquad \Phi = \int \frac{1}{2\gamma} \mathbf{C}^2 d\tau,$$

the integrations being extended throughout the whole fluid, which is supposed to extend to infinity, where all the hydrodynamic vectors vanish. In order to avoid the introduction of surface integrals referring to the surface of the fluid bodies, the surface of every fluid body is supposed to be a very thin layer through which the hydrodynamic quantities may vary with extreme rapidity, but yet continuously. At the outside of this surface layer the properties of a fluid body merge into those of the fundamental fluid. With this understanding the energy integral, (10), may be transformed as follows:

We may write

$$\tfrac{1}{2}\gamma \mathbf{c}_a^2 = \tfrac{1}{2}\gamma \mathbf{c}_a \cdot \mathbf{c}_a = \tfrac{1}{2}\mathbf{C} \cdot \mathbf{c}_a,$$

and we may express the vector $\mathbf{C}$ as the sum of a lamellar and a solenoidal part, thus

$$\mathbf{C} = -\gamma \nabla \varphi + \operatorname{curl} \mathbf{G},$$

wherein φ is a scalar, and $\mathbf{G}$ a vector potential; and we may now write

$$\tfrac{1}{2}\gamma \mathbf{c}_a^2 = -\tfrac{1}{2}\mathbf{C} \cdot \nabla \varphi + \tfrac{1}{2}\mathbf{c}_a \cdot \operatorname{curl} \mathbf{G}.$$

Substituting this value in equation (10) we get

$$\Phi = -\int \tfrac{1}{2}\mathbf{C} \cdot \nabla \varphi d\tau + \int \tfrac{1}{2}\mathbf{c}_a \cdot \operatorname{curl} \mathbf{G} d\tau,$$

which after performing simple vector transformations becomes

$$\Phi = \int \tfrac{1}{2}\varphi \operatorname{div} \mathbf{C} d\tau + \int \tfrac{1}{2}\mathbf{G} \cdot \operatorname{curl} \mathbf{c}_a d\tau. \tag{12}$$

It will now be shown that if throughout the fluid system

$$\operatorname{div} \mathbf{C} = 0,$$

$$\operatorname{curl} \mathbf{c} = 0,$$

$$\mathbf{C}_e = 0,$$

then the hydrodynamic field must vanish throughout the system. To do this we first notice that if $\mathbf{C}_e = 0$, then $\mathbf{c}_e$ must also vanish, and that

$$\operatorname{curl} \mathbf{c} = \operatorname{curl} \mathbf{c}_a - \operatorname{curl} \mathbf{c}_e;$$

and since $\mathbf{c}_e = 0$ and $\operatorname{curl} \mathbf{c} = 0$, that $\operatorname{curl} \mathbf{c}_a$ must vanish. Now if $\operatorname{curl} \mathbf{c}_a$ and $\operatorname{div} \mathbf{C}$ both vanish throughout the system, the total energy, Φ, must also vanish, as is evident from equation (12). The expression for Φ given in equation (11) must also vanish, and since the integral in this expression involves only positive terms, it follows that $\mathbf{C}$ must vanish everywhere. Now since $\mathbf{C}$ and $\mathbf{C}_e$ must both vanish everywhere under the given conditions, $\mathbf{C}_i (= \mathbf{C} - \mathbf{C}_e)$ must also vanish everywhere. Finally, since $\mathbf{C}$, $\mathbf{C}_i$ and $\mathbf{C}_e$ are necessarily zero at all points of the system, it is evident that the field must vanish completely.

It is now easy to show that any two fields which have the same distribution for the divergence of the specific momentum, the energetic specific momentum, and the curl of the induced velocity, can-

not differ in other respects, and that therefore when these quantities are given throughout the fluid the field is uniquely determined.

To do this it is only necessary to consider a third field, the descriptive vectors of which are the differences of corresponding vectors of the two fields considered. It is evident that this third field will be one which has no divergence of the specific momentum, no energetic specific momentum, and no curl of the induced velocity. Now we have just previously seen that such a field must vanish. Hence the two fields considered cannot differ; and it follows that the hydrodynamic field is uniquely determined by the distribution of the divergence of the specific momentum (= the local time rate of decrease of the density), of the energetic specific momentum, and of the curl of the induced velocity (= — the kinematic vortex density).

III. The Stationary Electric and the Stationary Magnetic Field.

In order to facilitate the comparison of the properties of the hydrodynamic with those of the stationary electric or the stationary magnetic field, a short summary of the properties of the stationary electric and of the stationary magnetic field will be given.

Electric and Magnetic Notation.—The notation employed to designate the actual, induced, and energetic vectors, and the electric and magnetic inductivities is given in Table II.

TABLE II.

Quantity.	Actual.		Induced.		Energetic.	
	Electric.	Magnetic.	Electric.	Magnetic.	Electric.	Magnetic.
Flux	$\mathbf{A}$	$\mathbf{B}$	$\mathbf{A}_i$	$\mathbf{B}_i$	$\mathbf{A}_e$	$\mathbf{B}_e$
Field intensity	$\mathbf{a}_a$	$\mathbf{b}_a$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{a}_e$	$\mathbf{b}_e$

Electric inductivity $= \alpha$. Magnetic inductivity $= \beta$.

Among the six electric vectors given in Table II. there are four independent relations, the equations of connection, which will be given presently; likewise for the magnetic vectors.

7. Geometric Properties of the Electric and Magnetic Field.

We will first give a summary of the geometrical properties of the stationary electric and the stationary magnetic field by writing down the appropriate equations which, according to Maxwell's theory, express our knowledge on this subject.

In the first place, there are two groups of equations, one for the electric and one for the magnetic case, which give the relationships among the various flux and field intensity vectors. These equations are referred to as the equations of connection, and are

$$(A_1)\qquad \begin{aligned} \mathbf{A} &= \mathbf{A}_i + \mathbf{A}_e, \\ \mathbf{a}_a &= \mathbf{a} + \mathbf{a}_e, \\ \mathbf{A}_i &= \alpha\mathbf{a}, \\ \mathbf{A}_e &= \alpha\mathbf{a}_e, \end{aligned} \qquad (B_1)\qquad \begin{aligned} \mathbf{B} &= \mathbf{B}_i + \mathbf{B}_e, \\ \mathbf{b}_a &= \mathbf{b} + \mathbf{b}_e, \\ \mathbf{B}_i &= \beta\mathbf{b}, \\ \mathbf{B}_e &= \beta\mathbf{b}_e. \end{aligned}$$

The equations which give the magnetic and electric current densities at any point in a stationary electric or stationary magnetic field are

$$(A_2)\qquad \operatorname{curl}\mathbf{a} = -\mathbf{i}, \qquad (B_2)\qquad \operatorname{curl}\mathbf{b} = \mathbf{j}.$$

The first of these equations states that the magnetic current density at any point in a stationary electric field is equal to the curl of the electric induced field intensity at the same point; $\mathbf{i}$ is a fictitious quantity, since the magnetic conduction current is unknown. The second equation states that the electric current density at any point in a stationary magnetic field is equal to the curl of the magnetic induced field intensity at the same point.

The volume densities of electricity and magnetism are expressed by the equations

$$(A_3)\qquad \operatorname{div}\mathbf{A} = E, \qquad (B_3)\qquad \operatorname{div}\mathbf{B} = M.$$

According to these equations, the divergence of the electric or the magnetic actual flux at any point represents the amount of electricity or of magnetism per unit volume at the same point. The latter quantity is fictitious, since true magnetism is unknown.

We have next to notice the equations which define the electric and magnetic properties of the free ether. They are

$$(A_4)\quad \begin{cases} (1)\quad \alpha = \alpha_0, \\ (2)\quad E = 0, \\ (3)\quad \mathbf{i} = 0, \\ (4)\quad \mathbf{A}_e = 0, \end{cases} \qquad (B_4)\quad \begin{cases} (1)\quad \beta = \beta_0, \\ (2)\quad M = 0, \\ (3)\quad \mathbf{j} = 0, \\ (4)\quad \mathbf{B}_e = 0. \end{cases}$$

8. *Dynamical Properties of the Electric and Magnetic Field.*

From the investigations of Heaviside it appears that the probable forms of expression for the mechanical forces exerted by an electromagnetic system against the exterior agencies acting upon it are, when reckoned per unit volume,

$$(A_5)\qquad \mathbf{f}_e' = \mathbf{a}_a \operatorname{div} \mathbf{A} - \tfrac{1}{2}\mathbf{a}_a^2 \nabla\alpha + (\operatorname{curl} \mathbf{a}_a) \times \mathbf{A},$$

$$(B_5)\qquad \mathbf{f}_m' = \mathbf{b}_a \operatorname{div} \mathbf{B} - \tfrac{1}{2}\mathbf{b}_a^2 \nabla\beta + (\operatorname{curl} \mathbf{b}_a) \times \mathbf{B},$$

due to the electric and magnetic field respectively. As these equations are identical in form, it will suffice to give the interpretation of the latter.

The first term on the right represents that part of the force which is due to true magnetism, and is probably fictitious since distributions of true magnetism are unknown. If such a force existed, it would be in magnitude equal to the product of the magnetic field intensity into the volume density of true magnetism ; and it would have the direction of the magnetic field intensity. The second term represents that part of the force which is due to heterogeneity of the medium and has the direction of the gradient of the magnetic inductivity. Since the magnetic energy per unit volume is $\frac{1}{2}\beta\mathbf{b}_a^2$, the magnitude of this force is the magnetic energy per unit volume multiplied by the ratio of the inductivity gradient to the inductivity itself. The force represented by the last term may be split into two distinct parts. For

$$\operatorname{curl} \mathbf{b}_a = \operatorname{curl} \mathbf{b} + \operatorname{curl} \mathbf{b}_e,$$

and hence

$$(\operatorname{curl} \mathbf{b}_a) \times \mathbf{B} = (\operatorname{curl} \mathbf{b}) \times \mathbf{B} + (\operatorname{curl} \mathbf{b}_e) \times \mathbf{B}.$$

Now curl $\mathbf{b}$ represents the *true* electric current density, and the term (curl $\mathbf{b}$) $\times$ $\mathbf{B}$ expresses the force due to the existence of *true* electric current. In magnitude it is equal to the product of the

magnitude of the *true* current density and the magnetic flux multiplied by the sine of the angle between the directions of these quantities; its line of action is perpendicular to the current and the magnetic flux and its direction is indicated by any one of the familiar rules. The term (curl $\mathbf{b}_a$) $\times$ $\mathbf{B}$ represents the force due to permanent magnetization, curl $\mathbf{b}_a$ being the fictitious current density which is employed on Ampere's theory in representing permanent magnetization. The magnitude and direction of this force are found as in the case for the force upon the true current.

By integration of equations (A_5) and (B_5) the expressions for the resultant forces upon an electrified or magnetized body in an electromagnetic field may be obtained. In this way we get, for the electric and the magnetic case respectively,

$$(A_6) \quad \mathbf{F}_e' = \int \mathbf{a}_a \operatorname{div} \mathbf{A} d\tau - \int \tfrac{1}{2} \mathbf{a}_a^2 \nabla \alpha d\tau + \int (\operatorname{curl} \mathbf{a}_a) \times \mathbf{A} d\tau,$$

$$(B_6) \quad \mathbf{F}_m' = \int \mathbf{b}_a \operatorname{div} \mathbf{B} d\tau - \int \tfrac{1}{2} \mathbf{b}_a^2 \nabla \beta d\tau + \int (\operatorname{curl} \mathbf{b}_a) \times \mathbf{B} d\tau,$$

the integrations extending throughout the whole body.

9. *Specifications which Uniquely Determine the Electric and Magnetic Fields.*

As is well known, a theorem analogous to that given in section (6) for the hydrodynamic field holds for the case of the stationary electric or the stationary magnetic field. Thus:

The electric field is uniquely determined when the distributions of the divergence of the actual electric flux, of the curl of the electric induced field intensity, and of the electric energetic flux, are given; and a similar statement holds for the magnetic field.

IV. Establishment of the Change-of-State Analogy.

In discussing the properties of the hydrodynamic system consisting of the *fluid bodies* and the *fundamental fluid*, we have made constant use of two fundamental vector quantities, viz., the velocity of the fluid and the specific momentum of the fluid. In order to reach the proper forms for the expressions giving the geometrical and dynamical properties of the fluid system, with reference to the analogy with the stationary electric or magnetic field, each of these

two fundamental vectors has been considered as made up of two others, the induced and energetic vectors. Six vector quantities in all have been employed, but only two of these are independent, as a glance at the equations of connection (C_1) will show. A knowledge of the distributions of any two of these six vectors is sufficient to characterize the state of the fluid system so far as its geometry is concerned. The dynamical properties of the fluid motion are to be inferred from the equation (C_5).

We have now to show that an analogy exists between the equations expressing the geometrical and dynamical properties of the hydrodynamic field and the equations defining the geometrical and dynamical properties of *either* the stationary electric field *or* the stationary magnetic field.

In establishing the analogies a table (Table III.) giving the scheme of correspondence of quantities and things will first be given, and then the justification of the scheme will be given. The notation employed to designate the various quantities is also given in Table III.

TABLE III.

SCHEME OF CORRESPONDENCE AND NOTATION.

Electric.	Hydrodynamic.	Magnetic.
The ether,	The fundamental fluid,	The ether.
Electrified bodies,	The fluid bodies,	Magnetized bodies.
Inductivity (a),	Density of fluid (γ),	Inductivity (β).
Flux ($\mathbf{A}$),	Specific momentum ($\mathbf{C}$),	Flux ($\mathbf{B}$).
Induced flux ($\mathbf{A}_i$),	Induced sp. momentum ($\mathbf{C}_i$),	Induced flux ($\mathbf{B}_i$).
Energetic flux ($\mathbf{A}_e$),	Energetic sp. momentum ($\mathbf{C}_e$),	Energetic flux ($\mathbf{B}_e$).
Field intensity ($\mathbf{a}_a$),	Velocity ($\mathbf{c}_a$),	Field intensity ($\mathbf{b}_a$).
Induced field intensity ($\mathbf{a}$),	Induced velocity ($\mathbf{c}$),	Induced field intensity ($\mathbf{b}$).
Energetic field intensity ($\mathbf{a}_e$),	Energetic velocity ($\mathbf{c}_e$,)	Energetic field intensity ($\mathbf{b}_e$).
Volume density of electricity (E),	Local time rate of decrease of density (H),	Volume density of magnetism (M).
Current density ($\mathbf{j}$),	Kinematic vortex density ($\mathbf{k}$),	Current density ($\mathbf{i}$).

The justification of the scheme of correspondence given in Table III. is to be found through comparison of the equations which express the properties of the hydrodynamic field of the change-of-state analogy with the corresponding equations of the stationary electric or magnetic field. We will collect and rewrite the equations which are to be compared.

The equations of connection, which give the relations among the actual, induced and energetic vectors, are:

	Electric.	Hydrodynamic.	Magnetic.
	$\mathbf{A} = \mathbf{A}_i + \mathbf{A}_e,$	$\mathbf{C} = \mathbf{C}_i + \mathbf{C}_e,$	$\mathbf{B} = \mathbf{B}_i + \mathbf{B}_e,$
(A_1) (C_1) (B_1)	$\mathbf{a} = \mathbf{a} + \mathbf{a}_e,$	$\mathbf{c}_a = \mathbf{c} + \mathbf{c}_e,$	$\mathbf{b}_a = \mathbf{b} + \mathbf{b}_e,$
	$\mathbf{A}_i = \alpha\mathbf{a},$	$\mathbf{C}_i = \gamma\mathbf{c},$	$\mathbf{B}_i = \beta\mathbf{b},$
	$\mathbf{A}_e = \alpha\mathbf{a}_e,$	$\mathbf{C}_e = \gamma\mathbf{c}_e,$	$\mathbf{B}_e = \beta\mathbf{b}_e.$

The equations which give the densities of the magnetic (fictitious) and the electric conduction currents, and the kinematic vortex density are:

Electric.	Hydrodynamic.	Magnetic.
(A_2) $\operatorname{curl}\mathbf{a} = -i,$	(C_2) $\operatorname{curl}\mathbf{c} = -\mathbf{k},$	(B_2) $\operatorname{curl}\mathbf{b} = \mathbf{j}.$

The equations which give the volume densities of true electricity, of true magnetism (fictitious), and the rate of decrease of the density of the fluid at any point are:

Electric.	Hydrodynamic.	Magnetic.
(A_3) $\operatorname{div}\mathbf{A} = E,$	(C_3) $\operatorname{div}\mathbf{C} = H,$	(B_3) $\operatorname{div}\mathbf{B} = M.$

The equations giving the properties of the free ether, and of the fundamental fluid are:

	Electric.	Hydrodynamic.	Magnetic.
	$\alpha = \alpha_0,$	$\gamma = \gamma_0,$	$\beta = \beta_0,$
(A_4) (C_4) (B_4)	$E = 0,$	$H = 0,$	$M = 0,$
	$\mathbf{i} = 0,$	$\mathbf{k} = 0,$	$\mathbf{j} = 0,$
	$\mathbf{A}_e = 0,$	$\mathbf{C}_e = 0,$	$\mathbf{B}_e = 0.$

The equations summarized above give the geometrical properties of the vectors which characterize the three types of fields.

The dynamical correspondence is established by referring to the equations which give the forces per unit volume exerted by the electric, magnetic, and hydrodynamic fields. The equations expressing these forces for the three cases are:

Electric.

$$(A_5) \qquad \mathbf{f}_e' = \mathbf{a}_a \operatorname{div}\mathbf{A} - \tfrac{1}{2}\mathbf{a}_a^2 \nabla\alpha + (\operatorname{curl}\mathbf{a}_a) \times \mathbf{A},$$

HYDRODYNAMIC.

$$(C_5) \qquad \mathbf{f}_h' = -\mathbf{c}_a \operatorname{div} \mathbf{C} + \tfrac{1}{2}\mathbf{c}_a^{\,2}\nabla\gamma - (\operatorname{curl} \mathbf{c}_a) \times \mathbf{C},$$

MAGNETIC.

$$(B_5) \qquad \mathbf{f}_m' = \mathbf{b}_a \operatorname{div} \mathbf{B} - \tfrac{1}{2}\mathbf{b}_a^{\,2}\nabla_f\beta + (\operatorname{curl} \mathbf{b}_a) \times \mathbf{B}.$$

The three fields are severally uniquely determined when the distributions of the following quantities are given in the three cases:

ELECTRIC.	HYDRODYNAMIC.	MAGNETIC.
div $\mathbf{A}$,	div $\mathbf{C}$,	div $\mathbf{B}$,
curl $\mathbf{a}$,	curl $\mathbf{c}$,	curl $\mathbf{b}$,
$\mathbf{A}_e$,	$\mathbf{C}_e$,	$\mathbf{B}_e$.

An inspection of the equations summarized above shows the degree of completeness of the analogy. It is quite complete except for the reversed algebraic signs in the expression for the force per unit volume in the hydrodynamic field, when compared with the corresponding expressions for the force in the electric or magnetic fields.

V. DISCUSSION OF THE ANALOGY.

The analogy which has been established shows that the vectors which are employed to describe the properties of the hydrodynamic field are geometrically related in the same way as the vectors which are used to describe the stationary electric or the stationary magnetic field. As far as the geometrical relationship of corresponding vectors is concerned the analogy is quite complete. The dynamical aspects of the analogy bring out a curious and interesting result; for, as we have just seen, the forces in the hydrodynamic field, although of the proper form, are oppositely directed to the corresponding forces in the electric or magnetic field. The same curious departure is present in the Bjerknes form of the analogy.

The correspondence of quantities set forth in Table III., has been justified — except for the opposite signs involved in the expressions found above for the corresponding forces.

In the Bjerknes form of the analogy the correspondence *at any instant* of the equations representing the properties of the hydrodynamic field with the corresponding equations for the electric or

magnetic field has just the same claim to completeness as that found in the change-of-state analogy. But owing to the fact that the *fluid bodies* in the Bjerknes form of the analogy always consist of the same particles of fluid, they must, in general, as time goes on necessarily change in form and in position in space.

If one attempts to make the *fluid bodies* in the Bjerknes form of the analogy persist in a fixed geometrical configuration, he may (following Professor Bjerknes) introduce the necessary conditions to insure a stationary state of motion of the fluid. This procedure, however, limits very much the generality of the motion, inasmuch as the possibility of motion normal to the surface of a *fluid body* is thereby excluded. The scope of the analogy is consequently much restricted.

With the same object in view Professor Bjerknes has also assumed a vibratory motion of the fluid, so that the *fluid bodies* shall execute small vibrations about invariable mean positions in space and thus *appear* to remain stationary. Here again a notable restriction upon the generality of the motion and upon the scope of the analogy is introduced. For in the Bjerknes form of the analogy the dynamic vortex density must be conserved at all points of the fluid, but this is the quantity which corresponds to electric, or magnetic, current density, and in a vibratory state of motion it is evident that it must vanish everywhere, since it cannot be varied. The hydrodynamic equations then reduce to those corresponding to *static* electric or magnetic fields with consequent loss of generality.

In the next section will be given the investigations of the conditions which have to be satisfied if the change-of-state analogy is to persist as time goes on. It will appear that no real loss in generality is involved through the introduction of these conditions; and that therefore the change-of-state analogy will persist in time and will hold for stationary as well as static electric or magnetic fields.

9. *Conditions for the Persistence of the Analogy.*

We have already seen that the state of motion in the hydrodynamic system is completely specified when the distributions of

$$\mathbf{C}_e, \text{ curl } \mathbf{c}, \text{ and div } \mathbf{C}$$

are given. Now in the stationary or steady state in the electric or

magnetic system the corresponding quantities are invariable with time; and if the analogy is to persist with time, it is necessary in the hydrodynamic case that

$$\frac{\partial \mathbf{C}_e}{\partial t} = 0,$$

$$\frac{\partial (\text{curl } \mathbf{c})}{\partial t} = 0,$$

$$\frac{\partial (\text{div } \mathbf{C})}{\partial t} = 0.$$

The condition has already been imposed (eq. (9)) that the energetic specific momentum shall be locally conserved; hence the first of these conditions must always be satisfied. The second condition requires that the curl of the induced velocity shall be locally conserved, or in other words that the density of the kinematic vortex shall be invariable with time. There is obviously no objection to imposing this condition. The third condition requires the local conservation of div $\mathbf{C}$ or, on account of the equation of continuity, (2), that $\partial r/\partial t$ shall be constant with respect to the time. Owing to the introduction of this condition some difficulty of interpretation appears. For at points where $\partial r/\partial t$ has other than zero value, we have to conceive the density as always increasing or always decreasing. If it be supposed that the initial density at such points is so great that finite variations of its value may be neglected, the difficulty disappears. Or, we may restrict the analogy, so that it applies to those electric and magnetic systems only which have no distribution of true electricity or magnetism. In the case of the magnetic field we have reason to believe that true magnetism does not exist. In the light of the difficulty just discussed it would appear that the analogy more appropriately applies to the stationary magnetic field.

The difficulty might also be overcome through the introduction of a state of vibratory motion into the fluid system, in a manner similar to that used by Professor V. Bjerknes, and referred to above. But this procedure has the disadvantage that the analogy is thereby restricted to static electric and magnetic fields.

Apart from the difficulty just discussed, there is nothing against

the introduction of the conditions for a steady state of motion in the hydrodynamic system, in order that the analogy shall persist with lapse of time. Perhaps the most characteristic feature of the change-of-state analogy is that which permits of the introduction of the conditions for a steady state of motion in the fluid system without requiring the motion normal to the surface of a *fluid body* to vanish, with the consequent loss of generality in the analogy.

VI. Another Possible Form of the Analogy.

Another form of the change-of-state analogy may be obtained by adopting a different mode of development of the hydrodynamic equation of motion than that given in detail above. It does not appear, however, to furnish an analogy which is any broader than the one already dealt with. For this reason it will suffice to refer briefly to its principal features only, omitting the steps in the mathematical development. The correspondence of quantities is the same as in the first form of the analogy.

Following a process very similar in many respects to that employed in developing the equation of motion in the first form of the analogy, it appears that the hydrodynamic equation of motion,

$$\gamma \frac{d\mathbf{c}_a}{dt} = \mathbf{f} - \nabla p, \tag{1}$$

may be written in the form

$$\frac{\partial \mathbf{C}_i}{\partial t} + \nabla(\tfrac{1}{2}\gamma \mathbf{c}^2 + \mathbf{C}_e \cdot \mathbf{c}) - \tfrac{1}{2}\mathbf{c}^2 \nabla\gamma + (\operatorname{div} \mathbf{C})\mathbf{c} + (\operatorname{curl} \mathbf{c}) \times \mathbf{C} - \mathbf{c}\mathbf{C}_e \nabla = \mathbf{f} - \nabla p. \tag{4'}$$

The induced specific momentum is now required to satisfy the condition

$$\frac{\partial \mathbf{C}_i}{\partial t} = -\nabla(p + \tfrac{1}{2}\gamma \mathbf{c}^2 + \mathbf{C}_e \cdot \mathbf{c}), \tag{5'}$$

and the energetic velocity must then satisfy the equation

$$\gamma \frac{d\mathbf{c}_e}{dt} = \mathbf{f} - \mathbf{c}(\operatorname{div} \mathbf{C}) + \tfrac{1}{2}\mathbf{c}^2 \nabla\gamma - (\operatorname{curl} \mathbf{c}) \times \mathbf{C} + \mathbf{c}\mathbf{C}_e \nabla. \tag{6'}$$

Equations (5′) and (6′) should be compared with equations (5) and (6), the corresponding ones in the first form of the analogy.

From equation (5′) it is easily proved that the dynamic vortex density must be conserved. Furthermore it may readily be shown that the geometrical properties expressed by the equations $(C_1)\cdots(C_4)$ are also characteristic of the second form of the analogy.

In order to arrive at the dynamic analogy the condition that the energetic velocity of every fluid particle be conserved is introduced, so that

$$(9') \qquad \frac{d\mathbf{c}_e}{dt} = 0.$$

This equation should be compared with equation (9), the corresponding one in the first form of the analogy.

The introduction of the condition (9′) leads to the following expression for the elementary force exerted per unit volume of fluid against exterior agencies:

$$(C_5') \qquad \mathbf{f}_h'' = -\mathbf{c}\,(\operatorname{div}\mathbf{C}) + \tfrac{1}{2}\mathbf{c}^2\nabla\gamma - (\operatorname{curl}\mathbf{c}) \times \mathbf{C} + \mathbf{c}\mathbf{C}_e\nabla.$$

This equation should be compared with equation (C_5). It will be noticed that the expression for the elementary force in equation (C_5') does not correspond in form to the corresponding expressions for the electric or magnetic field as given by Heaviside, and expressed in equations (A_5) and (B_5). Granted that the equations of Heaviside referred to are the correct ones, it is evident that the first form of the analogy is much to be preferred. It is only just to remark, however, that the same form of expression for the resultant force acting upon a *fluid body* is obtained through volume integration of the equation (C_5'), as that obtained from equation (C_5) by the same process.

The first form of the analogy possesses some advantages over the second when the conditions for a steady state of motion are introduced, arising from the difference in the conditions (9) and (9′).

Professor Bjerknes has shown in his lectures delivered at Columbia University that, with the correspondence of quantities which he assumes, there are at least two possible forms of the analogy. The first of these formally corresponds to the first form of the change-of-state analogy; and the second to the second form of the change-of-state analogy, outlined above.

SUMMARY.

The object of the foregoing paper is to present the main features of a new form of a hydrodynamic analogy with the stationary electric or magnetic field.

The analogy developed is termed the change-of-state analogy, and is a modification of the well-known Bjerknes analogy. For this reason there is given at the beginning of the paper a short account of the principal features of the Bjerknes analogy. A general description of the hydrodynamic system employed in the change-of-state analogy is then given. The differences in the two forms of analogies arise, primarily, from the different correspondence of the hydrodynamic with the electric or magnetic quantities which is assumed.

The correspondence of quantities which is assumed in the change-of-state is the following :

HYDRODYNAMIC.	ELECTRIC OR MAGNETIC.
Velocity	Field Intensity
Specific momentum	Flux
Density	Inductivity.

In the Bjerknes form of the analogy velocity corresponds to flux, specific momentum to field intensity, and specific volume to inductivity.

The hydrodynamic system of the change-of-state analogy consists of a *fundamental fluid* of infinite extent, within which there are certain fixed regions, called *fluid bodies*. The density of the *fundamental fluid* is supposed to be constant both in time and space. The fluid in a *fluid body* may be compressible, and the density variable from point to point. In both cases a perfect fluid is assumed.

The chief distinguishing feature of the fluid system is to be found in the conditions which allow a fluid body to remain fixed in space and yet permit of motion of the fluid, both within and without, normal to its surface. Thus the *fundamental fluid* may enter the region of a fluid body, experiencing as it does so, however, a change of state, acquiring the density and properties of motion of the *fluid body*, and losing those of the fundamental fluid. In like

manner, fluid may leave the region of a *fluid body*, at the same time acquiring the properties of the *fundamental fluid*, and losing those of the *fluid body*. This feature permits of the assumption of a steady state of motion of the fluid without serious loss of generality in the scope of the analogy.

A summary of the most important results found is given below:

1. The *fundamental fluid* may be consistently assigned properties which are analogous to those of the free ether.

2. The *fluid bodies* may be consistently assigned properties which are analogous to those of the stationary electric or the stationary magnetic field — with the exception that the mechanical forces to which they are subject are oppositely directed to the corresponding forces acting upon material bodies in the electric or magnetic field.

3. The geometrical relationships among the velocity and momentum vectors and the density of the fluid is expressed by a system of equations which are of the same type as the equations of Maxwell, expressing the relationships among the field-intensity and flux vectors and the inductivity in the stationary electric or the stationary magnetic field.

4. The mechanical force acting per unit volume of fluid is expressed in the form of an equation which is of the same type as the equation given by Heaviside, expressing the mechanical force acting per unit volume in the electric or magnetic field. This means, of course, that the resultant force acting upon a whole *fluid body* must correspond, except for the reversed sign, to the resultant force acting upon a material body in the electric or magnetic field.

5. The analogy established permits of the assignment of a steady state of motion to the hydrodynamic system; consequently the analogy is one which will persist with lapse of time, and without loss of generality. In this respect it possesses an advantage over the Bjerknes form of the analogy.

6. A second form of the change-of-state analogy is shown to be possible. The same correspondence of quantities is employed as in that to which the above results refer. The second form of the analogy leads to an expression for the mechanical force, acting per unit volume of the fluid, which is different from that found in the first form. It does not correspond therefore to the equation given

by Heaviside for the corresponding case in the electric or magnetic field. But the expression for the *resultant* force upon a *fluid body* is the same as in the first form of the analogy, and therefore corresponds to the expression as given by Heaviside for the corresponding electric or magnetic case.

COLUMBIA UNIVERSITY,
January, 1908.

A MECHANICAL EFFECT ACCOMPANYING MAGNETIZATION.

BY O. W. RICHARDSON.

ON the electron theory of matter the magnetic properties of bodies arise from the motion of the constituent electrons of their atoms. It is well known that all bodies may be divided according to their magnetic quality into three classes: (1) diamagnetic, (2) paramagnetic and (3) ferromagnetic. Bodies belong to the diamagnetic class if their atoms possess no resultant magnetic axis. They would also belong to this class even if they possessed a resultant magnetic axis provided this axis had no tendency to set in a definite direction in an external magnetic field. By the word atom in this discussion is meant a magnetic atom; that is, the smallest portion of the matter under consideration whose electronic properties are such that the magnetic properties of the whole can be built up by the piling together suitably of similar particles. The facts appear to indicate that in many cases the magnetic atoms are the same as the chemical atoms, in other cases possibly not.

Paramagnetic and ferromagnetic substances both appear to result when the constituent atoms both possess a resultant magnetic axis and also set themselves in a definite direction under the influence of an external magnetic field. The difference between the two classes appears to depend on the relative magnitude of the atomic magnetic moment and the forces called into play by the displacement of the atoms. When the restoring forces of non-magnetic type due to a small displacement give rise to a couple which is great compared with that exerted by the external field on the equivalent atomic magnet, the magnetic fields at our disposal will only produce small rotations of the atoms and these small rotations will be proportional to the external magnetic field. For small rotations the *resultant* intensity of magnetization will be proportional to the rotation. Thus the intensity of magnetization will always be proportional to

the magnetizing field over the range of values of the latter which are practicable; the body will thus be paramagnetic.

Ferromagnetic substances are those for which the restoring couples are not great compared with those of magnetic type. In this case we are able to apply external fields great enough to bring all the atomic magnets into line with the applied field. The body will then possess its maximum or "saturation" value of intensity of magnetization. For a fuller discussion of this aspect of the subject the reader is referred to a paper by Langevin.[1]

On this view it will be seen that the distinction between paramagnetic and ferromagnetic substances is one of degree only. It depends only on the magnitude of the restoring forces of non-magnetic type compared with the displacing forces of magnetic type. It might be possible for a substance which was in general paramagnetic to become ferromagnetic by combining it with some substance which increased the freedom of motion of its atoms. This may be the explanation of the peculiar behavior of the alloys of manganese investigated by Heusler. It is obvious that the converse case is also to be expected and may be applied to explain the behavior of some of the alloys of iron and the other ferromagnetic metals.

On the view that we have taken the resultant magnetic fields which the atoms of magnetic (as opposed to diamagnetic) substances possess, arise from the motion of their constituent electrons in closed orbits. If e is the charge in electromagnetic units carried by an electron describing an orbit possessing approximately circular symmetry and lying approximately in one plane, then if a is the area of the orbit and t the time in which it is described it can be shown that the value of the magnetic force, at any point whose distance is considerable compared with the dimensions of the orbit, averaged over a revolution is the same as that arising from a magnet of moment ea/t in electromagnetic units. The equivalent magnet passes through the center of the orbit and is perpendicular to the plane of the latter. The component of the magnetic moment in any assigned direction is given by the formula ea_p/t where a_p is the resolved part of a perpendicular to the direction in question. If there are n of the above orbits per unit volume then the component, in the direction con-

[1] Journal de Physique, Vol. IV, p. 678, 1905.

sidered, of the magnetic moment of the element of volume dv, supposed to contain a large number of them, is $nea_p/t \cdot dv$.

If the azimuths of the different orbits differ then the average value of a_p must be taken in the above formula. In the case of an unmagnetized bar of iron one direction is as probable as another for the azimuth so that the average value of a_p is equal to zero.

If we proceed to calculate the intensity of magnetization induced in the substance by a uniform external field in the direction of z, taking account of the possible presence of both positive and negative electrons in motion, we find

$$M_z = NE\left[\frac{\overline{A_p}}{T}\right] + ne\left[\frac{\overline{a_p}}{t}\right]. \qquad (1)$$

where the bars denote average values. Here M_z is the magnetic movement per unit volume, and for the right hand side the capitals refer to the positive ions. In this formula the proper signs must be given to E and e and also to $\overline{A_p}$ and $\overline{a_p}$ according to the sign of the charge and the direction of rotation. If there are only rotating negative electrons present then M_z has the value given previously, namely, nea_p/t.

We shall now proceed to obtain an expression for the moment of momentum of the revolving electrons about the direction of the applied magnetic force. Let the line about which the moment of momentum is calculated be the axis of z so that it is given by the equations $x = y = 0$. Consider any approximately circular orbit, the coördinates of whose center are $x_0 y_0 z_0$. Let the coördinates of the revolving electron referred to this center at any instant be ξ, η, ζ; thus its coördinates referred to the point on the axis about which the moment of momentum is to be taken are $x_0 + \xi$, $y_0 + \eta$, $z_0 + \zeta$. These are the displacements of the particle, and its velocity components are $d\xi/dt$, $d\eta/dt$ and $d\zeta/dt$. The moment of momentum about the z-axis is thus

$$m\left(\overline{x_0 + \xi}\,\frac{d\eta}{dt} - \overline{y_0 + \eta}\,\frac{d\xi}{dt}\right) = m\left\{x_0\frac{d\eta}{dt} - y_0\frac{d\xi}{dt} + \xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right\}.$$

Averaging this over a complete revolution evidently the mean values

are

$$x_0\overline{\frac{d\eta}{dt}} = 0, \quad y_0\overline{\frac{d\xi}{dt}} = 0$$

and

$$\overline{\xi\frac{d\eta}{dt}} = -\overline{\eta\frac{d\xi}{dt}} = \frac{\text{area of projected orbit}}{t} = a_p/t.$$

Hence the average moment of momentum about any axis is independent of the position of that axis, so long as its direction is the same and is equal to $2ma_p/t$. If there are revolving electrons with both positive and negative charges we find, with the same convention as to signs as that already used, that U_z the moment of momentum per unit volume arising from the motion of the electrons is given by

$$U_z = 2\left\{NM\left[\overline{\frac{A_p}{T}}\right] + nm\left[\overline{\frac{a_p}{t}}\right]\right\}. \qquad (2)$$

We may regard the formulæ (1) and (2) as embracing all the electrons in the atom whether moving or not. In this case since the body as a whole is uncharged we have $NE = ne$. The quantities A_p/T and a_p/t will represent the resolved areal velocities of all the positive and negative electrons respectively; we may write then for brevity A and a respectively. With this understanding

$$M_z = nea\left[1 + \frac{A}{a}\right]$$

and

$$U_z = 2nma\left[1 + \frac{e}{E}\frac{M}{m}\frac{A}{a}\right] = 2\frac{m}{e}M_z\frac{1 + \frac{e}{E}\frac{M}{m}\frac{A}{a}}{1 + \frac{A}{a}}. \qquad (3)$$

For small fields A and a will both be proportional to the applied field so that A/a will be a constant independent of the field strength and depending only on the constitution of the atom. So that in every case the moment of momentum per unit volume will be proportional to the intensity of magnetization, the factor of proportionality depending only on the charges and masses of the electrons and their configuration in the atom.

The most usual form of the electron theory of matter assumes

that the negative electrons alone are in motion and most of the experimental facts seem to be in favor of this conclusion. In this case the above result acquires an important simplification. If the positive electrons are free from orbital motions $A = 0$ so that

$$U_z = 2\frac{m}{e}M_z. \tag{4}$$

Thus the ratio between the moment of momentum per unit volume and the intensity of magnetization is the same for all substances and is equal to twice the inverse of the specific charge (e/m) of the negative electrons.

Experiments are at present in progress in the physical laboratory of Princeton University with the object of detecting the existence of this moment of momentum experimentally. Consider a long thin cylindrical bar of iron suspended by a fiber passing through its axis of figure so that it is capable of vibrating about a vertical axis. When the bar is not magnetized its constituent electrons will not possess a resultant moment of momentum about any axis as on the average one azimuth is as probable as another for the orbits. Now consider the effect of suddenly applying a vertical magnetic field. The movable orbits will set so as to leave a balance in favor of the plane perpendicular to the direction of the field. There will thus be created a moment of momentum about the axis of suspension. But by the laws of dynamics the total moment of momentum of any self-contained system is invariable. The moment of momentum acquired by the revolving electrons must thus be balanced by an equal reaction elsewhere. It would appear that this reaction is to be sought for in one of two places. The most reasonable I think is to suppose that it is effective on the rest of the atom, the part which is not revolving in an orbit. In this case it would be made evident by a twisting of the suspended system as a whole. The twist would of course be purely temporary owing to the restoring effect of the torsional couple arising from the suspension. The other possibility is that there is a reaction of electromagnetic origin on the magnetizing system. Theoretically calculable reactions of this nature appear to be too small to account for the effect. In fact this explanation seems to be definitely excised by the fact that the

moment of momentum is proportional to the mass of the revolving particles and independent of their charges (except in so far as their mass is electromagnetic).

Assuming that the magnetism of iron is due solely to the motion of negative electrons the torsional reaction should easily give a measurable effect. If any appreciable portion of the magnetization were to arise from the motion of heavier charges the effect would be proportionately greater. It will be seen that this method forms a valuable addition to our present equipment for investigating magnetic phenomena. So far experimental success has not been attained owing to the difficulty of eliminating disturbing effects.

PRINCETON, N. J.,
December 23, 1907.

A NEW TYPE OF SEXTANT.

BY L. A. FREUDENBERGER.

THE ordinary sextant (Hadley's) makes use of the well known law of reflection — that the reflected and incident rays make equal angles with the normal to the mirror surface — in order to obtain a movement of a vernier arm of half the angle measured. The arc is graduated, however, to read the angle directly.

In the new type of sextant devised by the writer, the law of reflection, that the incident and reflected ray lie in the same plane, normal to the mirror surface, is taken advantage of to make the angular movement of the vernier-arm equal to the angle measured. The principle of the new type of sextant is substantially shown by Fig. 1.

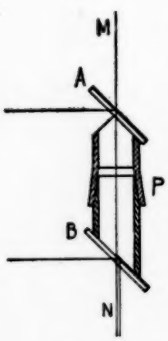

Fig. 1.

A is a full silvered mirror, mounted with adjusting screws to make an angle of 45° with the axis *MN* of the instrument. The mirror *A* rotates in the cone bearing *P* about the line of sight *MN* as an axis. A vernier-arm *C* (not shown in Fig. 1) is fastened to the frame of *A* and moves over a graduated arc fastened to the frame of *B*, serving to measure the angle of rotation of the mirror *A* around *MN* as an axis. *B* is a half silvered mirror mounted with adjusting screws to make an angle of 45° with the axis *MN*. In Fig. 1 the positions of the mirrors are shown for parallel rays (both mirrors reflecting an image of the horizon, which two images are brought in coincidence for the "zero" setting). If mirror *B* is kept sighted on the horizon, it is evident that with the sun at the meridian, mirror *A* will have to be rotated 90° about the axis *MN* in order to reflect the sun's image down the line of sight.

Figures 2 and 3 are photographs giving front and rear views of a sextant embodying the above principles, constructed by the writer, and in which the mirrors and accessories are clearly shown.

In the practical use of a sextant — sighting on the sun and the horizon, for instance — accuracy is limited by the difficulty the eye

Fig. 2.

experiences in deciding when the sun's disc just touches the horizon. In the Hadley sextant, suppose that the eye can determine the

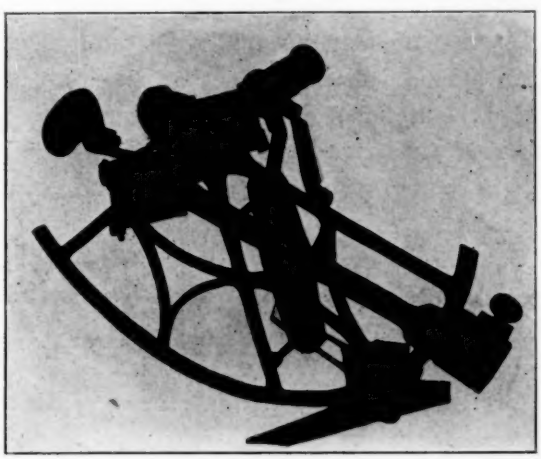

Fig. 3.

coincidence of two images to within one minute of angular movement of the mirror surfaces. This means that the actual angle be-

tween the two objects can only be determined within *two* minutes of angular measure. In the new type of sextant (Fig. 1), let the coincidence of two images be determinable again within one minute of angular movement of the mirror *A* in Fig. 1. This means that the actual angle between the two objects can also be determined within a minute of angular measure. This possibility of increasing the accuracy of measurement is, of course, the sole advantage of the new type of sextant. The new type of sextant differs from the old in the following particulars:

The field of view in the new type of sextant appears precisely the same as in the Hadley type, except that in driving forward the vernier-arm the moving field of view travels through a vertical arc of a circle, while the moving field of view in the Hadley sextant makes a straight vertical movement.

In locating two objects (see Fig. 2), so as to make them appear in the field of view, the line of sight of the telescope is at right angles to the plane containing the two objects and the observer, *i. e.*, the plane of the graduated arc of the instrument passes through the two objects. Vertical movements of the field of view are obtained by rotating the instrument as a whole about the line of sight of the telescope.

The Hadley type of sextant usually measures actual angles to 120° of arc or more. Since the angular movement of the mirror is only half this the sector of the graduated arc is only 60°, though graduated to read 120° directly. To secure an angular measurement of 120° in the new type of sextant, it is necessary to have a graduated sector of 120 actual degrees. Strictly speaking, the new instrument can hardly be called a sextant, since the graduated sector subtends one third of a circle instead of one sixth.

The adjustments of the mirrors in the new type of sextant can be accomplished most easily at sea where a horizon is always available.

To adjust for "zero" reading, the movable vernier-arm is clamped at the zero reading of the graduated arc. Mirror *A* (Fig. 1) is then adjusted until the two images of a very distant object appear in coincidence. To test whether the mirror *B* is properly adjusted, the two images should remain in coincidence independently of which part of the field of view is used. On sighting on a very distant

straight line (the horizon for instance), want of proper adjustment of mirror B would be shown by the two horizon lines, though meeting for the "zero" reading, not forming a continuous unbroken line but being bent at the line of division of the half silvered mirror. If desirable, the graduations and arc may be extended to 180° or even to 360°, when the adjustment of the mirrors may be exact, since the two horizon lines should coincide when the vernier-arm is set at 0° and also when set at 180°. An angle and its supplement might thus be measured, in order to compensate for eccentricity errors of the graduated arc.

DELAWARE COLLEGE,
NEWARK, DELAWARE

PROCEEDINGS

OF THE

AMERICAN PHYSICAL SOCIETY.

UPON THE MAGNETIC SEPARATION OF THE SPECTRAL LINES OF BARIUM, YTTRIUM, ZIRCONIUM AND OSMIUM.[1]

BY B. E. MOORE.

PRESTON[2] observed a similarity in the magnetic separation for spectral lines which formed similar series. Runge and Paschen[3] confirmed this observation for a number of substances. The inverse process would be to make use of the magnetic separation to investigate for series in substances where no such relationship has as yet been discovered. Runge used this method in barium, and found types of separation belonging to well-established series but not enough representatives to make a series. The present experiment proposes to use this inverse method in a more extended study of barium and other substances in which series have not been found.

The substances were volitilized in the usual way, by means of the spark in a strong magnetic field (24,400 cgs. units per sq. cm.) and photographed by means of a 21 ft. grating. The components parallel and perpendicular respectively to the lines of force were separated by a calcite prism. In barium there were observed three quadruplets of two different types, 4 sextets of three types, two octets of two types, 32 triplets which group themselves into four types and 21 triplets of various separations but generally represented in magnitude in some line having more than three components. The yttrium lines may be designated as follows: Two lines have 12 components each but are unlike; five lines have 9 components each but no duplicates; two lines have 8 components each, but two types; three lines are sextets, and two are duplicates; there are two dissimilar quintets; there are 14 quadruplets and possibly one duplication; there are 74 triplets whose separations do not collect around separations of well-defined magnitude so as to assemble them into groups as in barium.

[1] Abstract of a paper presented at the Chicago meeting of the Physical Society, December 30, 1907, to January 2, 1908.

[2] Phil. Mag. (5), 46, p. 325 (1898). Proc. Roy. Dub. Soc., VI., p. 385 (1898); VII., part II., p. 1.

[3] Berich. d. Berl. Akad. am 6 Feb., 1902; XIX., pp. 380, 720 (1902); XX., p. 417 (1904). Astrophys. Jour., XV., pp. 235, 333; XVI., p. 123.

Table of Intervals.

Interval or Approx. Part of 'a'	Multiples of the Intervals. Factors.	Number of Lines Represented.	Substance to which Types Belong.	Remarks.	λ
	22, 15, 12, 7	1	Zr		3459.1
	22, 16, 10, 9	1	Zr		4440.8
.10	13, 7	1	Zr		3891.61
$\left(\frac{a}{11}\right)$	15, 13, 11, 2, 0	1	Ne		
	20,16,14,12,8,8,4	1	Hg	1, NII [4]	3655
	12, 7	1	Os		4420.63
.12 $(a/9)^2$	7, 4	1	Y		5510.1
$\frac{a}{6}$ .18	7, 5, 4, 3, 2	1	Y		4235.89
$\frac{a}{6}$ .185	10, 9, 8, 7, 6, 2, 1, 0	1	Ne		
$\frac{a}{6}$ .19	14, 9, 5, 4, 0	1	Ne		
	9, 7, 5, 2, 0	1	Y		4398.21
.20	9, 6, 4, 3	1	Y		4083.89
	7, 6	1	Y		4682.5
$2a/11$	5, 1	1	Y		4375.11
	5, 2	1	Y		4177.68
	5, 4	1	Zr		4004.51
	10, 8, 7, 6, 4, 2	1	Hg		
.21	3, 2	1	Zr		3674.98
.22	9, 7, 3, 1	1	Y		4236.10
	4, 3	1	Zr		3894.00
$a/5$	4, 1	1	Zr		3916.16
	8, 7, 6, 5, 2, 1	1	Ne		
	8, 5, 4, 3		CuAg Al Tl	I. N. S.	
	8, 5, 4, 3	1	Ba	I. N. S.[1]	
	8, 4		Cu Ra Ag Tl	I. N. S.[1]	
.25	6, 5, 1, 0	1	Zr		4171.65
.26 $\left(\frac{2a}{9}\right)$?	6, 1	1	Zr	$3a \div 11$ [1,5]	4256.66
	6, 5, 3, 0	1	Zr	$\frac{a}{11}$ (15, 14, 8, 0)	4258.31
$\left(\frac{3a}{13}\right)^2$	8, 3, 2	3	Zr	$a \div 11$ (21, 8, 5)	See Zr 6 Comp.
	7, 2, 0	1	Zr	$a \div 11$ (18, 5, 0)	4061.70
$\frac{a}{4}$ .265	8, 5, 3, 2, 0	1	Y		3584.71
$\frac{a}{4}$ .27	6, 5	1	Zr	$a \div 10$ (16, 13)	3470.10
	6, 5, 4, 1, 0	1	Ne		
	7, 5, 4, 3	2	Zr	See Zr 8 Comp.	
	3, 2	1	Zr		4456.50

Table of Intervals. — Continued.

Interval or Approx. Part of '*a*.'	Multiples of the Intervals. Factors.	Number of Lines Represented.	Substance to which Types Belong.	Remarks.	λ
.29	7, 1	2	Zr	λ = 4003.28 and	3578.40
	2, 1	1	Zr		3588.15
.30	5, 2 (?)	1	Ba		5971.9
	4, 1	1	Y		3833.10
$\frac{3a}{11}$	4, 1	1	Zr		3890.58
	4, 3	1	Zr		3568.32
.305	6, 3, 2, 0	1	Zr	$a \div 11(18,9,7,0)$	4093.32
.32	9, 4, 0	1	Zr		4187.30
	5, 2	1	Zr		4418.80
$\frac{3a}{10}$ .33	4, 2	1	Zr		4034.30
	4, 1	1	Zr		4031.57
	2, 1	1	Zr		3284.89
.34	3, 2, 1	1	Zr		3396.51
	4, 1	1	Zr		3634.33
.36	5, 3, 1	1	Zr		3483.70
	3, 2, 2	1	Zr		3482.96
	5, 3, 1	2	Ba	H and	
	4, 2	2	Ba	2N	
$\frac{a}{3}$ .37	4, 2	1	Y		3200.44
	3, 1, 0	1	Zr		3376.42
	6, 4, 3, 2, 1, 0	2	Zr	See Zr 11 Comp.	
.38	5, 3, 1	1	Zr		3323.21
	5, 1, 0	1	Y		4477.1
	4, 1, 0	1	Y		3951.76
.39	4, 3, 2	1	Zr		4110.29
	4, 3	1	Zr		3434.08
$\left(\frac{4a}{11}\right)$	4, 3, 2, 1 (0?)		Hg	INII. h.	3125.8
.42	3, 2	8	Zr		
	3, 1	6	Zr		
$\frac{3a}{8}$	4, 1	3	Zr	See Zircon Quadruplet Table.	
	4, 2	2	Zr		
	2, 1	1	Zr		
.43	5, 4, 2, 1, 0	2	Zr	See Zr 9 Comp.	
	4, 3, 2, 0	1	Zr		3368.01
	6, 4, 1	1	Zr	$a \div 19\,(27, 17, 4)$	3554.31
	4, 3, 2, 2, 1, 1	1	Y		3818.49
.48	3, 1, 0	1	Zr		5350.5

Table of Intervals.—Continued.

Interval of Approx. Part of 'a.'	Multiples of the Intervals. Factors.	Number of Lines Represented.	Substance to which Types Belong.	Remarks.	λ
	5, 3 (1?)	1	Zr		3498.00
	3, 1	3	Zr	See Zr 4 Comp.	
5a/11	3, 2, 1, 1, 0	1	Y	[4]	3950.51
	3, 1	2	Y	See Y Quadruplet.	
.51	2, 1	1	Zr		4582.50
	3, 1	2	Zr	λ = 4429.28 and	4024.20
	3, 2 (?)	1	Zr		3480.59
.53	2, 1, 1, 0	1	Zr		4055.2
.54	5, 3, 2, 1, 0	1	Y	I. N. I. S.	4199.46
	4, 2, 2, 0, 0	1	Zr		4268.22
.55	3, 2, 1	1	Ba		
	3, 2, 1	2	Y	λ = 4358.91 and	3195.80
	2, 1	1	Y		
	3, 2, 1	1	Zr		4590.81
a/2	5, 3, 2	1	Zr		4431.70
	4, 3, 2, 1, 0	1	Ne		
	5, 3, 2, 1, 0		Hg	I. N. I. S.	
	5, 2, 1, 0		Hg	I. N. II. S.	
	4, 3, 2, 1		Hg	2NI	
	4, 3, 1		Hg	2NII	
	4		Hg	2NIII	
.57	2, 1	2	Zr	λ = 3934.99 and	3973.68
.60	3, 2, 2, 0	1	Zr		4214.05
6a ÷ 11	1, 2	1	Zr	[3]	3454.71
.63	3, 2, 1, 1, 0	1	Y		3628.89
3a/5 .67	3, 1, 0	1	Zr	a ÷ 11(20, 7, 0)	3501.50
(3a/4)83	2, 1, 0	1	Zr		4046.30

In zirconium, two lines have eleven components each and are duplicates, two lines have nine components and are duplicates, six lines have

[1] Ba 4166 is just as accurately represented by $a/4(6, 4, 3, 8)$ and Ra 4436, by $a/4(6, 3)$. Neither of the two systems represents it accurately. Multiples of $(a/11)$ represent these lines even better, except for one component.

[2] Series types, shown by Na, Cu, Ag, Al, Pt, Mg, Ca, Sr, Ba.

[3] The *p*-component lies outside of the *s*-component.

[4] When *p*- and *s*-components are duplicates in position, they are designated by repeating a number.

[5] The triplets of this first subordinate series are 3 and 5 times $a/5$.

eight components and two are duplicates, seven lines have seven components and all different types, sixteen lines have six components and not a duplicate, eleven lines have five components and likewise no duplicates, eighty lines are quadruplets. There are eight similar lines, when these are considered with reference to intensity. They subdivide into three groups, but even overlooking this no series could be found in these lines. There are six other lines of like separation but are irregularly distributed and therefore form no series. Four hundred and ten lines in zirconium are triplets. These lines have a wave retardation varying from $\pm .33$ to ± 2.75 per cm. Near these extremes there are few lines but the separations like the yttrium triplets do not collect in specific groups as was found in barium. Twenty-nine lines in zirconium do not show separation. Most of these lines are probably other types whose separation is too small, too weak or too diffused to be observed. In osmium one quadruplet and 31 triplets showed upon the photographic plates. In their irregularity the triplets of osmium behave like yttrium and zircon.

In comparing these lines with four or more components it became apparent that, although there were few duplicates, the magnitudes of separation were frequently repeated and that the separations of some lines could be picked out by taking one component from one line and another component from another line, etc. Combining some of these components together to form an ideal line with all components, it showed that these separations were nearly multiples of a smaller separation. This smallest distance between the components is designated the "interval," and the simple numbers by which they must be multiplied to produce the given separation the "factors." It was then found that the "intervals" were rather small in number for the substances studied and that the great variety in types arose from the variation in the "factors."

While engaged in this comparison an article appeared by Runge[1] bearing upon this same point. I have accordingly gathered a large part of Runge's material and my own in a table showing the character of the separations observed by both of us.[2] For the sake of clearness it may be said that Runge observed in his early observation that a triplet whose separation for my field strength would be 1.11 occurred very frequently and Runge designated this the "normal triplet." This separation corresponds to one of the four groups in barium. In yttrium, zirconium and osmium, the triplet separations would suggest that this "normal triplet" is purely arbitrary. This triplet separation gave Runge the

[1] Phys. Zeitschrift, No. 8, April 15, 1907.

[2] The observations of Miller and Purvis might have been introduced into this table also but this would only have encumbered the tables as the object of this table is to show that there are certain lines in Runge's substances giving the well-known series types, and that these types are not represented in the substances under study.

value of 1.75×10^7 for E/M (charge divided by mass) in the equation

$$a = \frac{\Delta\lambda}{\lambda^2} = \frac{E}{M}\left(\frac{H}{4\pi c}\right),$$

where c is the velocity of light and H the field strength.

Runge then makes the following statement: "Die bisher beobachteten pomplizierten Zerlegungen von spectrallinien in magnetischen Felde zeigen die folgende Eigentümlichkeit: Die Abstände der Komponenten von der Mitte sind Vielfache eines aliquoten Teils des normalen Abstandes 'a.' Sicher beobachtet sind bisher Teile $a/2$, $a/3$, $a/4$, $a/5$, $a/6$, $a/7$, $a/11$, $a/12$."

Under remarks in the table the abbreviations have the following meanings: N, subordinate series; S, satellite; H, principal series, *i. e.*, I. N. S. means first subordinate series satellite. The numbers refer to the special footnotes.

The most prominent feature of the table is the fact that most of the lines of the substances here studied do not belong to the types which are represented in the well-known series groups. The next prominent feature is that the types are seldom duplicated. Thirdly, great many types have common intervals, whose difference as before mentioned, consists in a variety of the factors by which the interval must be multiplied to produce the actual separation. Fourthly, whether or not the intervals are aliquot parts of a normal interval a is not so decisive. It can easily be contended that the intervals are irrational fractions. What is really needed is a thorough examination for a great variety of substances, to see if other lines with several components possess intervals of slightly different magnitude. Quadruplets are scarcely serviceable for this determination as the interval cannot be determined with sufficient accuracy. Eliminating the quadruplets from the present table the intervals are not far from rational and they may be designated as aliquot parts of Runge's normal triplet a.

Fifthly, the Zeemann triplets offer no great advantages for the study of series unless the magnitudes of the separations separate into well-defined groups, as they do not in zirconium, osmium and yttrium. Furthermore, as observed in barium, the triplet groups have separations noted in lines with more components, and these are all connected by simple intervals. In substances of the zirconium class, a triplet separation becomes the more confusing, because a definite separation may be a multiple of several intervals and a separation of practically the same value, within the limits of error, be multiples of other intervals. Hence there may be numerous types of about the same magnitude of separation. The method merely sifts the lines from which one may hunt series. Choosing all lines of about a certain magnitude of separation, then there may

be not more than 5 per cent. of the 410 in the group. These may be further subdivided by classification with reference to intensity. There then remains nothing to do but to try to fit an equation to the lines. Such a test means fatiguing labor as it requires three lines to determine the constants of the equation and a fourth line to test the accuracy. This method as applied to yttrium triplets was reasonably complete. In zircon only a few separations were tried. Both yielded no series.

Sixthly, series are eliminated from the lines having several components except under quadruplets of zircon designated by interval $3a/8$. No series was found here.

My thanks are extended to Professor Voigt, of the University of Göttingen, at whose suggestion this investigation was undertaken.

NOTE ON SELECTIVE REFLECTION AS A FUNCTION OF ATOMIC WEIGHT.[1]

BY W. W. COBLENTZ.

AFTER examining the reflection and transmission spectra, one or both, of about 300 substances, the observations lying in the region of the spectrum from 0.5 to 30+ μ the writer has found that it is an easy matter to work out all sorts of relations, only to learn, after gathering more data, that the whole thing is an illusion. It therefore seems that the linear relation between the weight of the element combined with equal amounts of oxygen in the acid radical found by Mr. L. B. Morse [2] by selecting certain maxima of reflection bands, is misleading. From the earliest work of Abney and Feasting to the latest (theoretical) work of Einstein, it has been generally accepted that oxygen is the active element in causing certain bands, just as sulphur has been found quite inactive. The great groups of chemically related compounds have been found to have similar absorption and reflection spectra, but heretofore no simple relation could be established between the spectra of the various groups. The present simple relation results from selecting particular reflection bands from the spectra of certain carbonates, sulphates, nitrates and silicates. The selection of KNO_3 (max. at 7.1 μ) from the nitrates by Morse seems arbitrary, for Pfund [3] has given a large number of nitrates which have a band in common at 7.45 μ. This seems to be a characteristic band of the nitrates, and $AgNO_3$ might have been selected instead of KNO_3. A characteristic band of the sulphates appears to be at 9.1 μ (harmonic with the one at 4.55 μ) while Morse selected a less frequent one at 8.6 μ. From the silicates, $MgSiO_3$, with an insignificant band at

[1] Abstract of a paper presented at the Chicago meeting of the Physical Soc ety, December 30, 1907, to January 2, 1908.

[2] Morse, Astrophys. Jour., 26, p. 241, 1907.

[3] Pfund, Astrophys. Jour., 24, p. 19, 1906.

9.1 μ, was selected, while Na_2SiO_3 with a sharper band at 9.9 μ, and $ZrSiO_4$ with a group of still more intense bands, at 10.1, 10.6 and 11 μ respectively, were not considered.

In $PbMoO_4$ the maxima lie in the region of 11.75 and 13 μ while in $CaWO_4$ there is a large reflection band with maxima at about 11.4, 11.9 and 12.5 μ respectively. These data including Morse's are plotted in

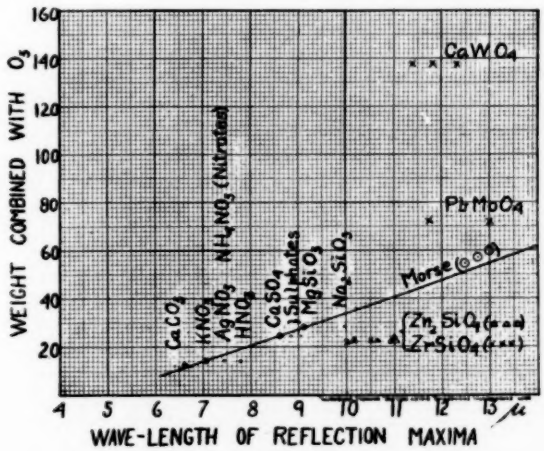

Fig. 1.

Fig. 1, and tabulated in Table I. from which it will be observed that the graph is not a straight line, and that if any relation exists it is a very complex curve, showing that for the region up to 10 μ the atomic weight

TABLE I.

Substance.	Chemical Formula.	Atomic Weight of Base.	Weight with 48 gr. of O.	Position of Band.
Calcite	$CaCO_3$	39.7	12 gr. of C	6.6 μ
Potassium nitrate	KNO_3	38.9	14 " " N	7.15 (7.05)
Anhydrite	$CaSO_4$	39.7	24 " " S	8.6
Enstatite	$MgSiO_3$	24.2	28 " " Si	9.1
Zircon	$ZrSiO_4$	90.6	28 " " Si	10, 10.6, 11 μ
Wulfenite	$PbMoO_4$	206.9	72 " " Mo	11.75, 13
Scheelite	$CaWO_4$	39.7	138 " " W	11.4, 11.9, 12.5 μ

of the element in the acid radical has a great effect in shifting the maximum, while beyond this point the atomic weight of the element united with oxygen is of minor importance. In fact the base and the element united with the oxygen in the acid radical seem to influence each other. As an illustration of the arbitrariness in selecting bands to establish

relations like the aforesaid, calcite, $CaCO_3$, may be noticed, in which the first reflection band is complex with maxima at 6.4 (?), 6.5, 6.6 and 7 μ while in its isomer, aragonite, the maxima are at 6.4 (?), 6.52, 6.74 and 7 (?) μ. In $SrCO_3$ the band with a maximum at 6.67 μ could not be resolved even with a fluorite prism. In crystalline quartz the maxima were found at 8.4 and 9.02 μ while in quartz glass the reflection bands are at 7.8, 8.4 and 8.8 μ respectively. Hence, one is at a loss to know what band to select from the carbonates to compare with the sulphates, nitrates and silicates. On the other hand, in the carbonates, and in the sulphates, the maxima of the bands have been plotted in their order of occurrence, which would seem to eliminate personal bias in the selection of maxima. From this it would appear that the simple, linear relation between the atomic weight of the base and the maxima of the reflection bands in the carbonates, and in the sulphates, is in agreement, at least as a first approximation. Even with this data at hand, speculation in regard to dynamical relations among atoms in the molecules had better be postponed until more data have been procured.

The data of Morse on the reflection bands of the carbonates at 11 μ to 15 μ are of interest in connection with the question of the value of the extinction coefficient necessary to give rise to selective reflection. The transmission spectrum of a thin section of calcite shows a small absorption band at 11.4 μ which coincides with the band found by reflection. It would seem desirable to examine the transmission of thin sections of carbonates in this region of the spectrum (using polarized rays) to compare with the intensity of the bands found by reflection.

In the region of the spectrum from 10 to 15 μ data can be obtained only under the greatest difficulties, and Morse's efforts have resulted in extending our knowledge throughout this region of the spectrum, verifying the previous work on the carbonates and the sulphates at 6 μ to 8 μ, and showing that the simple relation between the atomic weight of the base and the maxima of the reflection bands is in agreement, at least as a first approximation.

WASHINGTON, D. C.,
December 9, 1907.

THE USE OF COMPLEX QUANTITIES IN ALTERNATING CURRENTS.[1]

BY GEORGE W. PATTERSON.

IN the year 1797, a Danish surveyor, Caspar Wessal, presented to the Royal Academy of Sciences and Letters of Denmark a memoir *on the analytic representation of direction*,[2] in which he showed that a line

[1] Abstract of a paper presented at the Chicago meeting of the Physical Society, December 30, 1907, to January 2, 1908.

[2] Om Direktionens analytiske Betegning, et Forsög anvendt fornemmelig til plane og spaeriske Polygoners Oplösning, read March 10, 1797, published in Vol. V. of the

might be represented in magnitude and direction by expressions of the form

$$r(\cos v + \sqrt{-1}\sin v)$$

or

$$\varepsilon^{ma+mb\sqrt{-1}}$$

in which ε is the base of the Napierian system of logarithms, and r and ε^{ma} represent the length of the line, and v and mb the angle with the horizontal; and the expressions $\cos v + \sqrt{-1}\sin v$ and $\varepsilon^{mb\sqrt{-1}}$ are the analytic representation of the direction of the line. The memoir included many other facts concerning vectors, such as addition, subtraction, multiplication, division, powers and roots, which were subsequently rediscovered by later writers, as Wessel's paper suffered the fate of being put to sleep in the little-read published proceedings of the academy.

It is evident that if v or b are variable, the line will revolve with an angular velocity proportional to the rate of change of v or b. If we indicate by j the imaginary unit $\sqrt{-1}$, by t the time, by ω an angular velocity and by θ the epoch, we may represent the instantaneous direction and sense of the revolving vector by either of the equal expressions

$$\varepsilon^{j(\omega t+\theta)} = \cos(\omega t + \theta) + j\sin(\omega t + \theta).$$

It is to be noted that $\varepsilon^{j\frac{1}{2}\pi} = j$, and $\varepsilon^{-j\frac{1}{2}\pi} = -j$. The analytic representation of the vector in magnitude, direction and sense is expressed by either of the equal expressions

$$r\varepsilon^{j(\omega t+\theta)} = r(\cos(\omega t + \theta) + j\sin(\omega t + \theta)).$$

The real part of such an expression, if ω, θ and r are constant, may be used to represent a simple harmonic displacement, which may be defined as the projection of a uniform circular displacement upon a right line in the plane of the circle. But it is to be noted that this complex quantity as a whole represents uniform circular rather than simple harmonic displacement. In problems involving addition or subtraction the real part of the sum or the difference, respectively, will represent a simple harmonic sum or difference, and as a consequence will cause no difficulty in the interpretation of the result, for the real part of this sum or difference is the sum or difference of the real parts of the expressions to be added or subtracted; when, however, multiplication, division, powers or roots of such complex expressions are to be interpreted, an erroneous result

Memoirs of the Academy in 1799, and republished in French by the Academy, March 10, 1897. A review of this memoir is contained in Prof. W. W. Beman's vice-presidential address before section A of the American Association for the Advancement of Science at its Detroit meeting. Proc. A. A. A., 1897, Vol. XLVI., p. 33.

may follow, for the real part of the result is in part due to the imaginary parts of the factors. Such an error, for example, occurs when it is desired to multiply current by E.M.F. to get power, though no such error occurs when current and impedance are multiplied together to get E.M.F. Steinmetz tries to avoid error in the product of current and E.M.F. by introducing a very artificial rule to govern such multiplication.[1] Other writers abandon the process and simply write the correct answer.

It then appears evident that the use of a revolving vector to represent simple harmonic quantities, while convenient because of its simplicity in cases when no error results, is on the whole hazardous. It is the object of this paper to call attention to the use in simple harmonic problems of a vector-pair, by which I mean a pair of vectors equal in magnitude and symmetrically placed with respect to the axis of reals, in other words, a pair of conjugate complex quantities. If the components revolve, they must revolve in opposite directions with equal angular velocities. The resultant of a vector-pair is always real, *i. e.*, never complex or imaginary. The vector-pair in the case of uniform circular motion of its members has as a resultant a simple harmonic quantity. This is merely stating in another form the fact that two simultaneous circular motions in opposite sense, of the same magnitude as to amplitude and frequency and in the same plane, have as a resultant a simple harmonic motion of double amplitude, but of the same frequency and lying in the same plane. This stated analytically is

$$r(\varepsilon^{j(\omega t+\theta)} + \varepsilon^{-j(\omega t+\theta)}) = 2r \cos(\omega t + \theta).$$

This result might have been deduced from the exponential formula due to Euler for the cosine, which was well known before Wessel's day,

$$\cos\theta = \frac{\varepsilon^{j\theta} + \varepsilon^{-j\theta}}{2}.$$

In the case of an E.M.F. following a simple harmonic law with the epoch suitably chosen, we have the following equation:

$$e = E\cos(\omega t) = \frac{E}{2}(\varepsilon^{j\omega t} + \varepsilon^{-j\omega t}), \qquad (1)$$

which may be represented graphically as in Fig. 1.
A current following a similar law leads to a similar equation,

$$i = I\cos(\omega t - \theta) = \frac{I}{2}(\varepsilon^{j(\omega t-\theta)} + \varepsilon^{-j(\omega t-\theta)}), \qquad (2)$$

which has for its graphical representation Fig. 2.
The difference in phase between E.M.F. and current is represented by the

[1] C. P. Steinmetz, Alternating Current Phenomena, 3d edition, p. 151.

angle θ by which the current lags behind the E.M.F. (compare equations (1) and (2)). In both figures (1 and 2) the components of each

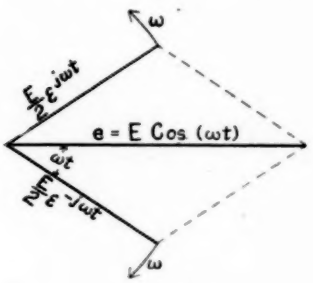

Fig. 1.

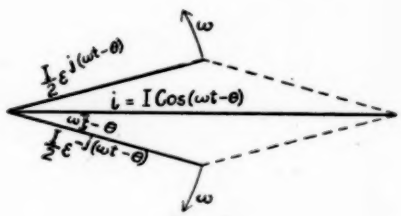

Fig. 2.

vector-pair revolve in opposite directions with angular velocity ω as indicated.

The equation connecting E.M.F. with resistance, current, inductance and time,

$$e = Ri + L\frac{di}{dt} = RI\cos(\omega t - \theta) + L\omega I\cos\left(\omega t - \theta + \frac{\pi}{2}\right),$$

may be thrown into the following form:

$$e = \frac{E}{2}(\varepsilon^{j\omega t} + \varepsilon^{-j\omega t}) = \frac{RI}{2}(\varepsilon^{j(\omega t-\theta)} + \varepsilon^{-j(\omega t-\theta)}) + \frac{L\omega I}{2}(\varepsilon^{j(\omega t-\theta+\frac{1}{2}\pi)} + \varepsilon^{-j(\omega t-\theta+\frac{1}{2}\pi)}). \tag{3}$$

Equation (3) may be graphically expressed by Fig. 3, in which triangles, similar to the usual E.M.F. triangles rotate in opposite directions. Each triangle has its separate equation.

Remembering that $\varepsilon^{j\frac{1}{2}\pi} = j$ and $\varepsilon^{-j\frac{1}{2}\pi} = -j$, we may represent the triangle revolving counter clockwise by

$$\frac{E}{2}\varepsilon^{j\omega t} = \frac{(R + jL\omega)I}{2}\varepsilon^{j(\omega t-\theta)} \tag{3'}$$

and that revolving clockwise by

$$\frac{E}{2}\varepsilon^{-j\omega t} = \frac{(R - jL\omega)I}{2}\varepsilon^{-j(\omega t-\theta)}. \tag{3''}$$

Combining equations (3') and (3'') we obtain

$$E^2 = (R^2 + L^2\omega^2)I^2$$

or

$$E = \sqrt{R^2 + L^2\omega^2}\, I. \qquad (3''')$$

The vector-pair is of most use in power problems, for it gives correct results without introducing any arbitrary rules. No other vector method, so far as I know, gives the correct expression for the *instantaneous* value of the power, though they all by *changing the rules for multiplication* force a correct answer for its *average* value. Taking as before for E.M.F. and current

$$e = E\cos(\omega t) = \frac{E}{2}(\varepsilon^{j\omega t} + \varepsilon^{-j\omega t}), \qquad (1)$$

$$i = I\cos(\omega t - \theta) = \frac{I}{2}(\varepsilon^{j(\omega t-\theta)} + \varepsilon^{-j(\omega t-\theta)}), \qquad (2)$$

and multiplying *without special rules*, we obtain for the power p at any instant

$$p = ei = EI\cos(\omega t)\cos(\omega t - \theta) = \frac{EI}{4}(\varepsilon^{j\theta} + \varepsilon^{-j\theta} + \varepsilon^{j(2\omega t-\theta)} + \varepsilon^{-j(2\omega t-\theta)}) = \frac{EI}{2}(\cos\theta + \cos(2\omega t - \theta)). \qquad (4)$$

Equation (4) brings out the important fact that power is a harmonic quantity of double the frequency of the current and E.M.F. It is, however, not simple-harmonic because the constant $\frac{1}{2}EI\cos\theta$ is added to the simple harmonic quantity of double frequency, $\frac{1}{2}EI\cos(2\omega t - \theta)$. Equation (4) is graphically represented by Fig. 4, which is composed of

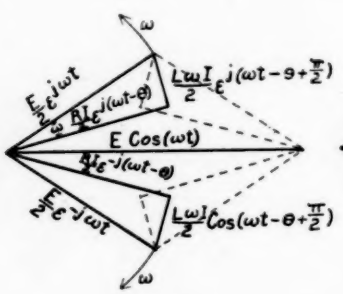

Fig. 3.

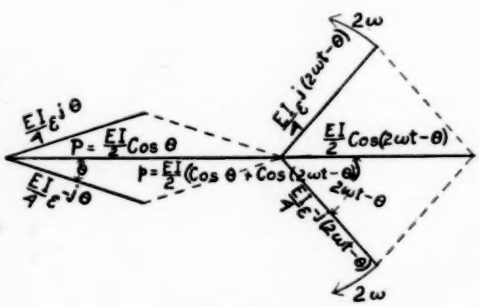

Fig. 4.

two vector-pairs, one stationary, and the other formed of a pair of vectors revolving in opposite senses with double angular velocity, 2ω. The constant resultant P of the stationary vector-pair represents the average value of the power as expressed in equation (5). The maximum value of the rotating vector-pair is $P/\cos\theta$, which is never less than P. It may equal P when the angle of lag θ becomes zero, which is the condition for a non-

inductive circuit. The instantaneous value of the power may be expressed also as

$$p = P + \frac{P}{\cos\theta}\cos(2\omega t - \theta).$$

A working model to represent the various phases of the power may be made by articulating a number of rods of equal length and rotating the movable components as indicated in Fig. 4.

In case that ω and θ are variable quantities, Figs. 1, 2, 3 and 4 will represent E.M.F. current, E.M.F. triangles and power for more complicated harmonic (not simple harmonic) quantities, but complications of such magnitude are beyond the scope of this paper.

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THE VARIATION OF APPARENT CAPACITY OF A CONDENSER WITH THE TIME OF DISCHARGE AND THE VARIATION OF CAPACITY WITH FREQUENCY IN ALTERNATING CURRENT MEASUREMENTS.[1]

BY BRUCE V. HILL.

I.

SEVERAL months ago, Professor Zeleny showed that the capacity of a condenser, as measured by the ballistic method, varied greatly with the time during which the condenser was connected with the galvanometer or with the period of the instrument. In his measurements, the time of connection was estimated from the distance over which the finger of the experimenter moved at a constant rate while depressing the charge-discharge key. The first part of the present study was undertaken to gain further information on this subject. In the second part it was desired to learn how the condensers used in ordinary telephone circuits might be expected to behave toward currents of the frequency of speech vibrations, whether they would show their rated capacities (measured with currents of probably not over 60 cycles) and whether they would distort the voice currents in any way other than that calculated from the capacity and the frequency of the various harmonics of the wave.

To measure very short intervals of time, four keys were mounted in pairs on two blocks. There was a make- and a break-circuit key in each pair. Instead of using a heavy pendulum to release the keys, as is usually done, they were mounted in a vertical position, one block below the other, and near them a $\frac{7}{8}$ inch cold drawn steel rod on which was a

[1] Abstract of a paper presented at the Chicago meeting of the Physical Society, December 30, 1907, to January 2, 1908.

sliding weight. The friction of the weight on the rod was first carefully determined. The distances between the triggers was measured by a cathetometer. In this way $\frac{1}{6000}$ of a second could be be read consistently. The galvanometer used was a Leeds and Northrup type P. d'Arsonval, weighted with two small bullets so that the period was 43 seconds.

Several condensers were tried: a Leeds and Northrup mica condenser, a Queen and Company's paper condenser and three telephone condensers which will be designated as *A*, *B* and *C*. The last was a tiny one of .25 M.F., intended for use in the base of a desk stand and had an insulation resistance so low that no measurements could be made with it. In each case the condenser was allowed to charge for a definite time and then discharged, the time between the end of the charging and the beginning of the discharge being only that required for the weight to fall a few centimeters. The time given for discharge varied from .0001 sec. to the quarter period of the galvanometer. These results are shown graphically together with the theoretical discharge curves. With the Mica condenser the apparent capacity was independent of the time of discharge. In the others the absorption is increasingly bad till, in the case of *B*, the discharge continued indefinitely.

II.

Though it is known that the capacity of a condenser, measured by means of an alternating current, varies with the frequency of the current used, the exact amount of this variation seems not to be very definitely known. The method finally employed was a modification of Anderson's arrangement, in a form in which it might readily be used for rapidly determining capacities with a degree of accuracy much greater than that usually attained in commercial work of this kind.

A small inductor generator was constructed, having an ebonite disk containing sixty soft iron inductors, rotating between the poles of the field magnet, on the ends of which were wound the armature coils. Frequencies up to 3,000 could thus be obtained, this being sufficiently beyond the range of ordinary voice currents. The same condensers were used as in the first part of the experiment. The Leeds and Northrup mica condenser fell in capacity 1 per cent. between 60 cycles and 3,000. Condenser *B* fell 2.55 per cent. in the same range. This decrease was so small that, for the present, no attention was paid to the effect of wave form. For all practical purposes, the decrease of capacity with higher frequencies is too small to be taken account of, though one writer in the *American Telephone Journal* claimed as high as 50 per cent.

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